On toric log schemes

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Abstract

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We define a restricted notion of flatness called *t-flatness* for an $\mathcal{O}_X$-module on an integral log scheme, investigate some of its basic properties, prove a Toric Flattening Theorem, and apply this theorem to establish the coherence of the structure sheaf on Kato's valuative log space. Thereafter, we replace the normality restriction from Kato's notion of log regularity with the weaker notion of torsion-freeness and show many results still hold in this relaxed context.

To the memory of Leo Thompson

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Contents

Introduction

A toric variety is a normal irreducible separated scheme X, locally of finite type over a field k, which contains an algebraic torus $T \cong (k^*)^d$ as an open set and is endowed with an algebraic action $T \times X \to X$ extending the group multiplication $T \times T \to T$. According to Oda [32]:

> The theory was started at the beginning of 1970's by Demazure [5] in connection with algebraic subgroups of the Cremona groups, by Mumford et al. [24] and Satake [35] in connection with compactifications of locally symmetric varieties, and by Miyake and Oda [27]. We were inspired by Hochster [13] as well as Sumihiro [39, 40].
>
> Comprehensive surveys from various different perspectives can be found in Danilov [3], Mumford et al. [24], [2] as well as [30, 31].

In [23], Kato extended the theory of toric geometry over a field to an absolute theory, without base. This is achieved by replacing the notion of a toroidal embedding introduced in [24] with the notion of a log structure. A toroidal embedding is a pair (X, U) consisting of a scheme X locally of finite type and an open subscheme $U \subset X$ such that (X, U) is isomorphic, locally in the étale topology, to a pair consisting of a toric variety and its algebraic torus. Toroidal embeddings are particularly nice locally Noetherian schemes with distinguished log structures.

A log structure on a scheme X, in the sense of Fontaine and Illusie, is a morphism of sheaves of monoids $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ restricting to an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$. The theory of log structures on schemes is developed by Kato in [21]. Log structures were developed to give a unified treatment of the various constructions of deRham complexes with logarithmic poles. In [17] Illusie recalls the question that motivated their definition:

> Let me briefly recall what the main motivating question was. Suppose S is the spectrum of a complete discrete valuation ring A, with closed (resp. generic) point s (resp. η), and X/S is a scheme with semi-stable reduction, which means

that, locally for the étale topology, X is isomorphic to the closed subscheme of $\mathbb{A}^n_S$ defined by the equation $x_1 \cdots x_n = t$, where $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$ and t is a uniformizing parameter of A. Then X is regular, X_η is smooth, and $Y = X_s$ is a divisor with normal crossings on X. In this situation, one can consider, with Hyodo, the relative deRham complex of X over S with logarithmic poles along Y,

$$\omega^{\cdot}_{X/S} = \Omega^{\cdot}_{X/S}(\log Y)$$

([14] see also [15, 16]). Its restriction to the generic fiber is the usual deRham complex $\Omega^{\cdot}_{X_\eta/\eta}$ and it induces on Y a complex

$$\omega^{\cdot}_Y = \mathcal{O}_Y \otimes \Omega^{\cdot}_{X/S}(\log Y).$$

One has $\omega^i_Y = \bigwedge^i \omega^1_Y$, and when X is defined as above by $x_1 \cdots x_r = t$, ω^1_Y is generated as an $\mathcal{O}_Y$-module by the images of the $d \log x_i$ $(1 \leq i \leq r)$ and dx_i $(r+1 \leq i \leq n)$ subject to the single relation $\sum_{1 \leq i \leq r} d \log x_i = 0$. The analogue of $\omega^{\cdot}_Y$ over $\mathbb{C}$ (S is replaced by a disc) is the complex studied by Steenbrink in [37], which "calculates" $\mathbf{R}\psi(\mathbb{C})$. If X/S is of relative dimension 1, ω^1_Y is simply the dualizing sheaf of Y (which probably explains the notation ω, chosen by Hyodo), and of course, in this case, $\omega^{\cdot}_Y$ depends only on Y. In general, however, $\omega^{\cdot}_Y$ depends not only on Y but also "a little bit" on X, so it is natural to ask: which extra structure on Y is needed to define $\omega^{\cdot}_Y$?

Assume for simplicity that Y has simple normal crossings, i.e. Y is a sum of smooth divisors Y_i meeting transversely. Let $\mathcal{L}_i$ be the invertible sheaf $\mathcal{O}_Y \otimes \mathcal{O}_X(-Y_i)$, and $s_i : \mathcal{L}_i \to \mathcal{O}_Y$ the map deduced by extension of scalars from the inclusion $\mathcal{O}_X(-Y_i) \hookrightarrow \mathcal{O}_X$. Then it is easily seen that the data consisting of the pairs $(\mathcal{L}_i, s_i)$, together with the isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \bigotimes \mathcal{L}_i$ (coming from $Y = \sum Y_i$), suffice to define ω^1_Y. Indeed, ω^1_Y can be defined as the $\mathcal{O}_Y$-module generated locally by elements $\{dx, d \log e_i\}$, for x a local section of $\mathcal{O}_Y$ and e_i a local generator of $\mathcal{L}_i$, subject to the usual relations among the dx, plus

$$s_i(e_i) d \log e_i = ds_i(e_i) \tag{0.1a}$$

$$\sum d \log e_i = 0 \text{ when } 1 \in \mathcal{O}_Y \text{ is written } \otimes e_i \tag{0.1b}$$

This construction and a subsequent axiomatic development was proposed by Deligne in [4], and independently by Faltings in [8]. In order to deal with the general case (Y no longer assumed to have simple normal crossings), it is more convenient to consider, instead of the pairs $(\mathcal{L}_i, s_i)$, the sheaf of monoids $\mathcal{M}$ on Y (multiplicatively) generated by $\mathcal{O}^*_Y$ and the e_i together with the map $\mathcal{M} \to \mathcal{O}_Y$ given by $\alpha(\otimes e_i^{n_i}) = \prod s_i(e_i)^{n_i}$ (replace $d \log e_i$ by $d \log e$ for $e \in \mathcal{M}$, and the relations (0.1) by $d \log(ef) = d \log e + d \log f$, $\alpha(e) d \log e = d\alpha(e)$, $d \log u = u^{-1} du$ if u is a unit). This is the origin of the notion of logarithmic structure, proposed by Fontaine and the speaker, which gave rise to the whole theory beautifully developed by Kato (and Hyodo) in [16, 20, 21, 22, 23].

Log structures are natural generalizations to arbitrary schemes of reduced normal crossings divisors on regular schemes. Here $\mathcal{M}$ is the sheaf of regular functions on the scheme that are invertible away from the divisor, which is a sheaf of monoids under multiplication. Log structures give rise to differentials with log poles, crystals and crystalline cohomology with log poles, and similar structures.

In [23], Kato defines a notion of log regularity and proves that fine saturated log regular schemes behave very much like toric varieties. If X is such a scheme, then X is normal and Cohen-Macaulay. In addition, Kato develops a theory of fans and subdivisions, shows how this theory can be used to resolve toric singularities, and identifies the dualizing complexes of such schemes.

In Chapter 5 we extend this theory by relaxing the requirement that the monoids be saturated, thereby relaxing the requirement that the schemes be normal. This is accomplished by casting the theory in terms of log structures alone. We do not appeal to the traditional theory of cones to keep track of the combinatorics since such an appeal forces the restriction to saturated monoids. Most of our results are local; that is, they are concerned with the singularities that occur on these schemes. We define the notion of a toric log regular scheme. Toric varieties are examples of such schemes, and toric log regular schemes behave like toric varieties in many ways.

In [28], Nakayama studies log étale cohomology. More properly, the paper studies Kummer log étale cohomology. The open sets are given by log étale morphisms of Kummer type. Every log étale map factors as a log blowup followed by a Kummer log étale map. In order to avoid the difficulties of working with log blowups, authors studying log étale cohomology restrict their attention to the Kummer log étale sites on fine saturated log schemes.

In Chapter 3 we will focus on log blowups, thereby complementing the present literature. In particular, much of this paper was motivated by the following question of W. Niziol: "If $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is a locally Noetherian fine log scheme, is the structure sheaf of Kato's valuative log space X^{val} coherent?" The assertion that the structure sheaf is universally coherent is Theorem 4.2.2. The proof presented here is based on notes prepared by O. Gabber.

In Chapter 1 we collect the algebraic preliminaries will we use in later chapters and the aspects of basic log geometry that we will emphasize. In Chapter 2 we define and study t-flatness. In Chapter 3 we prove a flattening theorem that is adapted to this

context. In Chapter 4 we recall the definition of Kato's valuative log space and use the flattening theorem to show its structure sheaf is coherent. In Chapter 5 we define log regularity and then generalize much of [23], relaxing the condition on the log structures from fine and saturated to toric. Although most of Kato's methods go through with only minor modifications, some extra lemmas are required.

Conventions and Notation

All monoids considered in this dissertation are commutative and cancellative. All rings considered in this paper are commutative and unital. See Kato [21] for an introduction to log schemes. There Kato defines pre-log structures and log structures on the étale site of X. However, we will use the Zariski topology throughout this dissertation.

P^*	the unit group of the monoid P.
$\overline{P}$	the sharp image of the monoid P, $\overline{P} = P/P^*$ is the orbit space under the natural action of P^* on P.
P^+	$P^+ = P \setminus P^*$ is the maximal ideal of the monoid P.
P^{gp}	the image of P under the left adjoint of the inclusion functor from Abelian groups to monoids.
$R[P]$	the monoid algebra of P over a ring R. The elements of $R[P]$ are written as "polynomials". That is, they are finite sums $\sum r_p t^p$ with coefficients in R and exponents in P.
$R[A]$	$\{\sum r_p t^p \mid p \in A\}$, where A is a subset of P^{gp}. If A is a P-set as well, $R[A]$ is an $R[P]$-module.
$R[A, B]$	$R[A]/R[B]$, where A and B are sub-P-sets of P^{gp} with $B \subseteq A$.
E/KE	$E/R[K]E$, where E is an $R[P]$-module and K is an ideal of P.
$I(K)$	the ideal $\beta(K)R$, where $\beta : P \to R$ is a monoid homomorphism with respect to multiplication on R and K is an ideal of P. We say such an ideal is a *log ideal* of R.

$R[[P]]$	the $I(P^+)$-adic completion of $R[P]$.
$P_\mathfrak{p}$	the localization of the monoid P at the prime ideal $\mathfrak{p} \subseteq P$.
$\dim P$	the (Krull) dimension of the monoid P.

Chapter 1

Preliminaries

See Kato [21] for an introduction to log schemes. There Kato defines pre-log structures and log structures on the étale site of X. However, we will use the Zariski topology throughout this dissertation. See Niziol [29] for a brief comparison of log structures on the Zariski and étale sites. In Kato [20], log structures on locally ringed spaces are defined and (for the most part) the Zariski topology is used.

1.1 Fine Log Scheme Basics

Definition 1.1.1. [20, Definition 1.2.3] Given a pre-log structure $\beta : \mathcal{M}_X \to \mathcal{O}_X$ on a scheme X, the *log structure associated to this pre-log structure* $\mathcal{M}^\alpha$ is defined to be the colimit of the diagram

$$\begin{array}{c} \beta^{-1}(\mathcal{O}_X^*) \to \mathcal{M}_X \\ \downarrow \\ \mathcal{O}_X^* \end{array}$$

in the category of sheaves of monoids on X, equipped with the homomorphism of sheaves of monoids induced by β and the inclusion $\mathcal{O}_X^* \subseteq \mathcal{O}_X$.

Remark 1.1.2. [21, Section 1.3] If $G \xleftarrow{\beta} P \xrightarrow{\gamma} Q$ is a diagram of monoids and G is a group, then the colimit of this diagram is $(G \oplus Q)/\sim$, where $\sim$ is the congruence given by $(g, q) \sim (g', q')$ if there exist $p, p' \in P$ such that $g+\beta(p) = g'+\beta(p')$ and $q+\gamma(p) = q'+\gamma(p')$. That is, (g, q) and (g', q') differ by an element of P^{gp}. In particular, we may replace P with any submonoid of $(\gamma^{gp})^{-1}(Q)$ that generates P^{gp} without changing the colimit.

Example 1.1.3. If P is a monoid, R is a ring, and $\beta : P \to R$ is a monoid homomorphism with respect to multiplication on R, then β induces a homomorphism of sheaves of monoids on $\operatorname{Spec} R$ from $P_{\operatorname{Spec} R}$ the constant sheaf on $\operatorname{Spec} R$ with stalk P to $\mathcal{O}_{\operatorname{Spec} R}$. We will denote the associated log scheme by $\operatorname{Spec}(P \xrightarrow{\beta} R)$.

A coherent log structure $\mathcal{M}$ on a scheme X is integral if and only if locally on X, $\mathcal{M}$ is isomorphic to the log structure associated to the pre-log structure $P_X \to \mathcal{O}_X$ for some finitely generated (cancellative) monoid P (see [20, Definition 1.2.6]).

Definition 1.1.4. If $\beta : P \to Q$ is a monoid homomorphism, we say β is *local* if $P^* = \beta^{-1}(Q^*)$. Furthermore, if R is a ring, $\mathfrak{p} \subset R$ is prime, and $\beta : P \to R$ is a monoid homomorphism with repect to multiplication on R, we say β is *local at* $\mathfrak{p}$ if the composite monoid homomorphism $P \xrightarrow{\beta} R \xrightarrow{\text{canonical}} R_{\mathfrak{p}}$ is local.

For the rest of this section, let P be a finitely generated monoid. Let R be a Noetherian ring and let $\beta : P \to R$ be a monoid homomorphism with respect to multiplication on R. Then $\operatorname{Spec}(P \xrightarrow{\beta} R)$ is a Noetherian fine log scheme. (In fact, every locally Noetherian fine log scheme is locally isomorphic to such a log scheme.) By Remark 1.1.2, $\operatorname{Spec}(P \xrightarrow{\beta} R)$ is isomorphic to $\operatorname{Spec}(Q \xrightarrow{\beta^{gp}|_Q} R)$ for any $Q \subseteq P + \beta^{-1}(R^*)$ containing P. After we prove the following proposition, we may assume β is local at any particular prime $\mathfrak{p} \subset R$ as well.

Proposition 1.1.5. *If $\mathfrak{p}$ is a prime ideal of R, then there exists an element $f \in R \setminus \mathfrak{p}$ and a finitely generated monoid $Q \subseteq P^{gp}$ containing P such that the map $Q \to R_f$ induced by β is local at $\mathfrak{p}$ and generates the same log structure on* $\operatorname{Spec} R_f$ *as β.*

Proof. Compose β with the canonical map $R \to R_{\mathfrak{p}}$ to get a map $\widetilde{\beta} : P \to R_{\mathfrak{p}}$. Let X be a finite generating set for P, let $X_0 = \{x \in X \mid \widetilde{\beta}(x) \in R_{\mathfrak{p}}^*\}$, let x_0 be the sum of the elements of X_0, let Q be the submonoid of P^{gp} generated by $X \cup \{-x_0\}$, and let $f = \beta(x_0)$. Now, $p \in P$ is mapped to a unit in $R_{\mathfrak{p}}$ if and only if p can be written as a sum of elements of X_0. Since the composition $P \xrightarrow{\beta} R \xrightarrow{\text{can.}} R_f$ maps x_0 to a unit, this composition factors through the localization $P \to Q$ by the universal property of localization. Since localization is an epimorphism, there is a unique map $Q \to R_f$ making the diagram

$$\begin{array}{ccc} P & \to & R \\ \downarrow & & \downarrow \\ Q & \to & R_f \end{array}$$

commute. By further localizing if necessary, we may assume the point $\mathfrak{p}$ is contained in every irreducible component of $\operatorname{Spec} R_f$. In particular, for every open subscheme U of $\operatorname{Spec} R_f$ containing $\mathfrak{p}$, we have $\Gamma(U, P_{\operatorname{Spec} R}) = P$ and $\Gamma(U, Q_{\operatorname{Spec} R_f}) = Q$. So, P and Q determine the same log structure near $\mathfrak{p}$. Now suppose q is an element of Q that is mapped into R_f^* by our map $Q \to R_f$. Since ring homomorphisms map units to units, q is sent to a unit by the map on the stalks $Q \to R_\mathfrak{p}$. Write $q = p - nx_0$ with $p \in P$ and $n \in \mathbb{N}$. Since q and x_0 are mapped to units in $R_\mathfrak{p}$, so is $p = q + nx_0$. Hence, p can be written as a sum of elements of X_0 and is therefore a unit in Q. That is, our map $Q \to R_f$ is local. $\square$

Remark 1.1.6. The log scheme $\operatorname{Spec}(k[x] \setminus \{0\} \xrightarrow{\text{inclusion}} k[x])$ is a quasi-coherent, integral log scheme such that at every point $\mathfrak{p}$, $\mathcal{M}_{\operatorname{Spec} k[x], \mathfrak{p}} \cong \mathbb{N}$. However, it is not fine. In fact, every open subscheme of $\operatorname{Spec} k[x]$ is of the form $\operatorname{Spec} k[x]_f$ where $f \in k[x]$ since $k[x]$ is a principal ideal domain. Since $k[x]$ contains infinitely many monic irreducible polynomials and f is only divisible by finitely many of them, the monoid $\overline{k[x]_f \setminus \{0\}}$ is infinitely generated for all $f \in k[x]$. So, there is no open neighborhood of any $\mathfrak{p} \neq 0$ such that the log structure on that open neighborhood is induced by a homomorphism from a finitely generated monoid, since that monoid would have to map onto $\overline{k[x]_f \setminus \{0\}}$.

1.2 Affine Semigroups

A theorem of Grillet [34, Theorem 3.11] says, "Let P be a finitely generated monoid. The monoid P is cancellative, reduced and torsion-free if and only if it is isomorphic to a submonoid of $\mathbb{N}^k$ for some positive integer k." We will strengthen this theorem by showing that such a monoid can be embedded in $\mathbb{N}^d$ with $d = \operatorname{rank} P^{gp}$ in such a way that a complete flag in P is taken to the standard flag in $\mathbb{N}^d$.

Definition 1.2.1. A monoid P is said to be *sharp* (or *reduced*) if its unit group is trivial.

A monoid P is said to be *torsion-free* if its difference group P^{gp} is torsion-free.

A monoid P is said to be an *affine semigroup* if it is isomorphic to a finitely generated submonoid of $\mathbb{N}^k$ for some $k \geq 0$.

A monoid S is said to be *saturated* (or *normal*) if for any positive integer n and any $p \in P^{gp}$, $np \in P$ implies $p \in P$. We call the smallest saturated submonoid of P^{gp} containing P the *saturation* of P and denote it by P^{sat}.

Proposition 1.2.2. *Let P be a monoid.*

(1) If P is sharp and saturated, then P is torsion-free.

(2) $\overline{P}$ is sharp.

(3) If P is saturated, then $\overline{P}$ is also saturated.

(4) If P is saturated, then every localization of P is also saturated.

(5) If P is finitely generated and saturated, then $P \cong P^* \bigoplus \overline{P}$.

Proof. (1) We will prove this by contradiction. Suppose p were a nontrivial torsion element of P^{gp} and suppose p had order n. Then we would have $np = 0$. So, p would be in P. But, $\mathbb{N}p$ would be a group. So, p would be a unit in P. This contradicts the fact that P is sharp.

(2) Suppose $\overline{p} \in \overline{P}^*$ and let $p \in P$ be mapped to $\overline{p}$ by the canonical map. Since p maps to a unit, there is an element $q \in P$ such that $p + q \in P^*$. Therefore, $q + (-(p + q))$ is the inverse of p and $p \in P^*$. Hence, $\overline{p} = 0$.

(3) Suppose $\overline{p} \in \overline{P}^{gp}$ and $n\overline{p} \in \overline{P}$ with $n \in \mathbb{N}$ positive. Let $p \in P^{gp}$ be a pre-image of $\overline{p}$. Since $n\overline{p} \in \overline{P}$, np differs from an element of P by an element of P^*. That is, $np \in P$. Since P is saturated, $p \in P$. Thus, $\overline{p} \in \overline{P}$.

(4) Let $Q = P - S$ with S a submonoid of P, let n be a positive integer, and let q be an element of $P^{gp} = Q^{gp}$ such that $nq \in Q$. Write $nq = p - s$ with $p \in P$ and $s \in S$. We have $n(q + s) = p + (n - 1)s$. So, $n(q + s) \in P$. Since P is saturated, $q + s \in P$. So, $q \in Q$.

(5) By (2), $\overline{P}$ is sharp. By (3), $\overline{P}$ is saturated since P is saturated. By (1), $\overline{P}$ is torsion-free. Furthermore, $\overline{P}^{gp}$ finitely generated and hence free. Therefore,

$$0 \to P^* \to P^{gp} \to \overline{P}^{gp} \to 0$$

is split exact. If $p \in P^{gp}$ is mapped into $\overline{P}$ by the righthand map, then p differs from an element of P by an element of P^*. Hence, $p \in P$ and the pre-image of $\overline{P}$ by the righthand map is P. That is, any section of $P^{gp} \to \overline{P}^{gp}$ maps $\overline{P}$ into P and $P \cong P^* \bigoplus \overline{P}$. $\square$

Remark 1.2.3. In Proposition 1.2.2, neither (3) nor (4) need be true if the word "saturated" is replaced with "torsion-free". Consider $P = \langle (1, 0), (1, 1), (0, 2), (0, -2) \rangle \subseteq \mathbb{Z}^2$, this monoid is finitely generated, saturated and torsion-free. However, $\overline{P} \cong \langle a, b \mid 2a = 2b \rangle$ is not saturated (since $a - b \notin \overline{P}$ and $2(a - b) = 0$) and $P \ncong P^* \bigoplus \overline{P}$ (since $\overline{P}$ is not torsion-free).

Consider $V = \mathbb{Q} \otimes_{\mathbb{Z}} P^{gp}$ the $\mathbb{Q}$-vector space generated by the monoid P. Let $C(X)$ be the cone over the subset $X \subseteq P$ in V, that is $C(X) = \{\sum_{i=1}^{n} q_i p_i \in V \mid \forall i, q_i \in \mathbb{Q}_{\geq 0},\ p_i \in$

$P\}$. If P is torsion-free, $P \to V$ is injective so we may freely identify P with its image in V. In this case, $P^{sat} = C(P) \cap P^{gp}$. In particular, if P is torsion-free, $C(P) = C(P^{sat})$.

Definition 1.2.4. Let P be a monoid. A submonoid $F \subseteq P$ is said to be a *face* of P if $p + p' \in F$ implies $p \in F$.

Notice that $F \subseteq P$ is a face if and only if $P \setminus F$ is a prime ideal. (We consider P to be a face and $\emptyset$ to be a prime ideal.)

Proposition 1.2.5. *Let P be a finitely generated, torsion-free monoid.*

(1) If F is a face of P, then $C(F)$ is a face of $C(P)$.

(2) If F is a face of $C(P)$, then $P \cap F$ is a face of P.

(3) If F is a face of P, then $P \cap C(F) = F$.

(4) If F is a face of $C(P)$, then $C(P \cap F) = F$.

This establishes a bijective correspondence between the faces of P and the faces of $C(P)$.

Proof. Evidently, whenever $X \subseteq P$ and $x \in C(X)$, there exists a positive integer n such that nx is in the monoid generated by X.

(1) Suppose x and y are elements of $C(P)$ such that $x + y \in C(F)$. Let n_1 be a positive integer such that $n_1 x \in P$, let n_2 be a positive integer such that $n_2 y \in P$, let n_3 be a positive integer such that $n_3(x + y) \in F$, and let m be the least common multiple of n_1, n_2 and n_3. We have $mx \in P$, $my \in P$ and $mx + my = m(x + y) \in F$. Since F is a face of P, $mx \in F$ and $my \in F$. So, $x \in C(F)$ and $y \in C(F)$.

(2) Suppose p and p' are elements of P such that $p + p' \in P \cap F$. Since F is a face, both p and p' are in F. So, $p \in P \cap F$ and $p' \in P \cap F$.

(3) Suppose p is an element of $P \cap C(F)$. Let n be a positive integer such that $np \in F$. Since $(n - 1)p + p \in F$ and F is a face, $p \in F$. Evidently, $F \subseteq P \cap C(F)$.

(4) If $x \in C(P \cap F)$, then for some positive integer n, $nx \in P \cap F$. In particular, $nx \in F$. Since F is a face, $x \in F$. If $x \in F$, then for some positive integer n, $nx \in P \cap F$. In particular, $nx \in C(P \cap F)$. Since $C(P \cap F)$ is a face, $x \in C(P \cap F)$. $\square$

In light of this, when P is finitely generated and torsion-free, we freely speak of edges (resp. facets etc.) meaning faces whose corresponding faces in $C(P)$ are edges (resp. facets etc.).

Definition 1.2.6. A sequence of faces $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ in a monoid P is said to be a *flag* in P. A flag in P is said to be *complete* if $F_0 = P^*$, $F_r = P$ and whenever F is a face of P lying between F_{i-1} and F_i, $F = F_{i-1}$ or $F = F_i$.

The complements in P of the faces in a flag form a chain of prime ideals. If the flag is complete, the chain of primes is saturated; that is, there is no prime properly between consecutive primes of the chain. Furthermore, if $\varphi : P \to Q$ is a homomorphism of monoids and G is a face of Q, $\varphi^{-1}(G)$ is a face of P. We will say φ takes the flag $G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r$ in Q to the flag $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ in P via pullback if $\varphi^{-1}(G_i) = F_i$ for all $0 \le i \le r$.

Let $\mathbf{e}_i \in \mathbb{N}^d$ be the element with a 1 in the ith position and zeroes elsewhere. We call $\{0\} \subset \langle \mathbf{e}_1 \rangle \subset \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \subset \cdots \subset \mathbb{N}^d$ the standard flag in $\mathbb{N}^d$.

Theorem 1.2.7. *Let S be a sharp finitely generated torsion-free monoid and let $\{0\} = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d$ be a complete flag in P. Then there exists an inclusion $\varphi : P \to \mathbb{N}^d$ taking the standard flag in $\mathbb{N}^d$ to the given flag in P and inducing an isomorphism $\varphi^{gp} : P^{gp} \to \mathbb{Z}^d$.*

Proof. Without loss of generality we may assume P is saturated, since the faces of P correspond bijectively to the faces of $C(P)$ according to Proposition 1.2.5. We proceed by induction on d. When $d = 0$, the theorem is trivial.

Suppose we have a map $\widetilde{\varphi} : F_{d-1} \to \mathbb{N}^{d-1}$ that sends the flag $\{0\} = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{d-1}$ to the standard flag in $\mathbb{N}^{d-1}$ and inducing an isomorphism $\widetilde{\varphi}^{gp} : F_{d-1}^{gp} \to \mathbb{Z}^{d-1}$. Let σ be a splitting of the inclusion $F_{d-1}^{gp} \to P^{gp}$ such a splitting σ exists since F_{d-1} is a face and P is saturated by Proposition 1.2.2. Now let $\psi_1 : P \to \mathbb{Z}^{d-1}$ be the map

$$P \to P^{gp} \xrightarrow{\sigma} F_{d-1}^{gp} \to \mathbb{Z}^{d-1}.$$

That is, inclusion into P^{gp} followed by σ followed by the isomorphism $\widetilde{\varphi}^{gp}$. Notice that if $p \in F_{d-1}$, then the coordinates of $\psi_1(p)$ are non-negative.

Let $\mathfrak{p}$ be the complement of F_{d-1}. Since F_{d-1} is a facet of P, $\mathfrak{p}$ is a height one prime of P. $\overline{P_{\mathfrak{p}}} \cong \mathbb{N}$ since P was assumed to be saturated. Let $\psi_2 : P \to \mathbb{N}$ be be the map

$$P \to P_{\mathfrak{p}} \to \overline{P_{\mathfrak{p}}} \cong \mathbb{N}.$$

Notice that $\psi_2(p) = 0$ if and only if $p \in F_{d-1}$. Now I claim $\psi = (\psi_1, \psi_2) : P \to \mathbb{Z}^{d-1} \times \mathbb{N}$ is injective. If $\psi(p) = \psi(p')$, then their last coordinates are equal and $p - p' \in F_{d-1}^{gp}$. Since the

map $F^{gp}_{d-1} \to \mathbb{Z}^{d-1}$ above is an isomorphism, ψ is injective. Furthermore, since some element of P maps to an element of $\mathbb{Z}^d$ whose last coordinate is one and the map $F^{gp}_{d-1} \to \mathbb{Z}^{d-1}$ above is an isomorphism, ψ^{gp} is an isomorphism.

Recall that the unique minimal generating set of an affine semigroup is called its Hilbert basis, see Sturmfels [38, Chapter 13]. For every element p of the Hilbert basis of P, let $(p_1, p_2, \ldots, p_d) = \psi(p)$. For $1 \le i \le d$, choose $n_i \in \mathbb{N}$ such that $p_i + n_i p_d \ge 0$ for every Hilbert basis element $p \in P$, and let θ be the automorphism of $\mathbb{Z}^d$ given by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & n_1 \\ 0 & 1 & 0 & \cdots & 0 & n_2 \\ 0 & 0 & 1 & \cdots & 0 & n_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & n_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Take $\varphi = \theta \circ \psi$. $\square$

Example 1.2.8. Let $P = \langle (0,2), (1,0), (2,-2) \rangle \subset \mathbb{N}^2$ and consider the flag $\{0\} \subset \langle (0,2) \rangle \subset P$. Suppose we already have $\widetilde{\varphi} : \langle (0,2) \rangle \to \mathbb{N}$, given by $(0,2) \mapsto 1$. Let $\psi_1 : P \to \mathbb{Z}$ be given by $(0,2) \mapsto 1$, $(1,0) \mapsto 0$ and $(2,-2) \mapsto -1$. $\psi_2 : P \to \mathbb{N}$ is given by $(0,2) \mapsto 0$, $(1,0) \mapsto 1$ and $(2,-2) \mapsto 2$. Therefore, $\psi : P \to \mathbb{Z} \times \mathbb{N}$ is given by $(0,2) \mapsto (1,0)$, $(1,0) \mapsto (0,1)$ and $(2,-2) \mapsto (-1,2)$. Let $n_1 = 1$, then

$$\theta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

So, φ is given by $(0,2) \mapsto (1,0)$, $(1,0) \mapsto (1,1)$ and $(2,-2) \mapsto (1,2)$.

Definition 1.2.9. We say a monoid V is *valuative* if, for every $v \in V^{gp}$, $v \in V$ or $-v \in V$. We say a monoid Q is an *overmonoid* of a monoid P if $P \subseteq Q \subseteq P^{gp}$. We say a monoid Γ is an *totally ordered monoid* if Γ is equipped with a total order $<$ such that if $\gamma, \gamma', \gamma'' \in \Gamma$ and $\gamma < \gamma'$, then $\gamma + \gamma'' < \gamma' + \gamma''$.

If P is a monoid, we define the divisibility pre-order $\le$ on P as follows: For $p, p' \in P$, we say $p \le p'$ if there exists a $q \in P$ such that $p + q = p'$.

Proposition 1.2.10. *Let P be a monoid.*

(1) The divisibility pre-order on P is a pre-order.

(2) The divisibility pre-order on P is a partial order if and only if P is sharp.

(3) The divisibility pre-order on P is a total order if and only if P is sharp and valuative.

(4) If P is valuative, then P is saturated.

(5) If $\varphi : P \to V'$ is a monoid homomorphism whose codomain is valuative, then φ factors through a valuative overmonoid of P.

Proof. (1) We need to show $\leq$ is reflexive and transitive. It is reflexive since $p + 0 = p$ for all $p \in P$. It is transitive because $p+q = p'$ and $p'+q' = p''$ together imply $p+(q+q') = p''$.

(2) We need to show $\leq$ is antisymmetric if and only if P is sharp. First, let's prove we have a partial order when P is sharp. Suppose $p \leq p'$ and $p' \leq p$. Then, there exist $q, q' \in P$ such that $p + q = p'$ and $p' + q' = p$. In particular, $p + q + q' = p$. Since our monoids are cancellative, $q + q' = 0$. Since P is sharp, $q = q' = 0$ and $p = p'$. On the other hand, if P is not sharp let p be a nontrivial unit of P. We have $0+p = p$ and $p+(-p) = 0$. So, $0 \leq p$ and $p \leq 0$ but, $p \neq 0$.

(3) It suffices to prove that P is valuative if and only if, for any pair $p, p' \in P$, $p \leq p'$ or $p' \leq p$. But, in light of the fact that every element of P^{gp} can be written as a difference of two elements of P, this is just a restatement of the valuative property: It says $p' - p \in P$ or $-(p' - p) = p - p' \in P$.

(4) Suppose $p \in P^{gp}$ and $np \in P$ for some positive integer n. Since P is valuative, $p \in P$ or $-p \in P$. If $p \in P$, we are done. If not, $-p \in P$. But, if $-p \in P$, then $p = (n-1)(-p) + np$ is in P.

(5) Evidently, φ factors through the overmonoid $V = \{p \in P^{gp} \mid \varphi^{gp}(p) \in V'\}$. So, it suffices to prove V is valuative. Suppose $v \in V^{gp}$ and $v \notin V$, we want to show $-v \in V$. From the definition of V, we know $\varphi^{gp}(v) \notin V'$. So, $-\varphi^{gp}(v) = \varphi^{gp}(-v) \in V'$. That is, $-v \in V$. $\square$

1.3 Toriodal Log Schemes

Definition 1.3.1. We say a log structure $\mathcal{M}$ on X is *torsion-free* if $\overline{\mathcal{M}}_X$ is a sheaf of torsion-free monoids. We say a log structure $\mathcal{M}$ on X is *toriodal* if it is fine and torsion-free. We say a log scheme $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is *toriodal* if $\mathcal{M}_X$ is toroidal.

Warning 1.3.2. If P is a sharp finitely generated torsion-free monoid, R is a Noetherian ring, and $\beta : P \to R$ is amonoid homomorphism with respect to multiplication on R, $\operatorname{Spec}(P \xrightarrow{\beta} R)$ need not be toroidal. Let k be a field and consider $\beta : \langle(1,0),(1,1),(0,2)\rangle \to k[x,y,z]/(x^2z - y^2)$ given by $(1,0) \mapsto x$, $(1,1) \mapsto y$, and $(0,2) \mapsto z$. Here $\overline{M}_{(x,y)} \cong \langle a, b \mid 2a = 2b\rangle$. See Remark 1.2.3.

Proposition 1.3.3. *If $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X) = \operatorname{Spec}(P \xrightarrow{\beta} R)$ is a fine log scheme and $\beta : P \to R$ is local at $x = \mathfrak{p}$, then $\overline{\mathcal{M}}_{X,x} \cong \overline{P}$. Furthermore, if $0 \to P^* \to P^{gp} \to \overline{P}^{gp} \to 0$ is split exact, then there is a monoid homomorphism $\sigma : \overline{P} \to P$ such that the morphism $\operatorname{Spec}(\overline{P} \xrightarrow{\sigma\circ\beta} R) \to \operatorname{Spec}(P \xrightarrow{\beta} R)$ induced by σ is an isomorphism.*

Proof. Since the question of whether $\overline{\mathcal{M}}_{X,x}$ is isomorphic to $\overline{P}$ is local, we may assume x is contained in every irreducible component of X. In particular, we may assume any neighborhood U of x is connected and $\Gamma(U, P_X) = P$. Let $\overline{\beta} : P \to R_{\mathfrak{p}}$ be the composition of β and the canonical map $R \to R_{\mathfrak{p}}$. By Remark 1.1.2, $\mathcal{M}_{X,x} \cong R^*_{\mathfrak{p}} \oplus P/\{\overline{\beta}(u)^{-1} \oplus u \mid u \in P^*\}$. We may now take the quotient $\mathcal{M}_{X,x}/\mathcal{M}^*_{X,x}$ to form $\overline{\mathcal{M}}_{X,x}$. Evidently, $\overline{\mathcal{M}}_{X,x} \cong (R^*_{\mathfrak{p}} \oplus P)/(R^*_{\mathfrak{p}} \oplus P^*)$. But, $\overline{P} \cong (R^*_{\mathfrak{p}} \oplus P)/(R^*_{\mathfrak{p}} \oplus P^*)$.

Consider the exact sequence of groups

$$0 \to P^* \to P^{gp} \to \overline{P}^{gp} \to 0.$$

If $p \in P^{gp}$ is mapped into $\overline{P}$ by the righthand map, then p differs from an element of P by an element of P^*. Hence, $p \in P$ and the pre-image of $\overline{P}$ by the righthand map is P. Let $\sigma : \overline{P} \to P$ be the restriction to $\overline{P}$ of a splitting of our exact sequence. By Proposition 1.1.5, for each point $\mathfrak{q}$ of $\operatorname{Spec} R$, there is a localization Q of $P \cong P^* \oplus \overline{P}$ and an open neighborhood $\operatorname{Spec} R_f$ of $\mathfrak{q}$ such that $Q \to R_f$ generates the same log structure on $\operatorname{Spec} R_f$ and is local at $\mathfrak{q}$. Since $P \cong P^* \oplus \overline{P}$, there exists a localization Q' of $\overline{P}$ such that $Q \cong P^* \oplus Q'$. Let $\widetilde{\beta} : Q \to R_{\mathfrak{q}}$ be the composition of our map $Q \to R_f$ with the canonical map $R_f \to R_{\mathfrak{q}}$, and let $\widetilde{\sigma} : Q' \to R_{\mathfrak{q}}$ be the similar map induced by $\beta \circ \sigma$. As above, $\mathcal{M}_{X,\mathfrak{q}} \cong R^*_{\mathfrak{q}} \oplus Q/\{\widetilde{\beta}(u)^{-1} \oplus u \mid u \in Q^*\}$. But, since $Q \cong P^* \oplus Q'$, we may first take the quotient by $\{\widetilde{\beta}(u)^{-1} \oplus u \mid u \in P^*\}$. That is, $\mathcal{M}_{X,\mathfrak{q}} \cong R^*_{\mathfrak{q}} \oplus Q'/\{\widetilde{\sigma}(u)^{-1} \oplus u \mid u \in (Q')^*\}$. So, the morphism of log schemes $\operatorname{Spec}(\overline{P} \xrightarrow{\sigma\circ\beta} R) \to \operatorname{Spec}(P \xrightarrow{\beta} R)$ induced by σ is an isomorphism. □

Corollary 1.3.4. *Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a fine log scheme and let x be a point on X such that $\overline{\mathcal{M}}_{X,x}$ is a torsion-free monoid. Then, there exists an open neighborhood U of x*

and a homomorphism $\beta : \overline{\mathcal{M}}_{X,x} \to \mathcal{O}_X(U)$ such that β induces $\alpha|_U$ and the composition of the induced map $\overline{\mathcal{M}}_{X,x} \to \mathcal{M}_X(U)$ with restriction to the stalk $\mathcal{M}_{X,x}$ and the canonical map $\mathcal{M}_{X,x} \to \overline{\mathcal{M}}_{X,x}$ is the identity.

1.4 The Prime Filtration Theorem

Theorem 1.4.1. *(Prime Filtration Theorem) Let P be a finitely generated torsion-free monoid, let $R = \bigoplus_{p \in P} R_p$ be a Noetherian P-graded ring, and let $E = \bigoplus_{p \in P^{gp}} E_p$ be a finitely generated P^{gp}-graded R-module. Then, E has a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

with each $E_i/E_{i-1} \cong R/\mathfrak{p}_i$ for some homogeneous prime ideal $\mathfrak{p}_i \subset R$.

Proof. By the usual prime filtration theorem [7, Proposition 3.7], E has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

with each $E_i/E_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. By induction on n, it suffices to prove that $\mathfrak{p}_1$ is homogeneous.

Since $R/\mathfrak{p}_1 \subseteq E$, $\mathfrak{p}_1$ is an associated prime of E. Hence, there is a least positive integer s such that there exists an element e of E with annihilator $\mathfrak{p}$ and P^{gp}-homogeneous elements e_i of E such that $e = \sum_{i=1}^{s} e_i$. By the results of § 3 of Gilmer [10], there is a total order $\prec$ compatible with the group structure on P^{gp} since P^{gp} is torsion-free. For each i, let q_i be the degree of e_i. We may assume $q_1 \prec q_2 \prec \ldots \prec q_s$. Let r be an element of $\mathfrak{p}_1$. Write $r = \sum_{i=1}^{m} r_i$ with each r_i P-homogeneous of degree p_i and $p_1 \prec p_2 \prec \ldots \prec p_m$. We want to show that each homogeneous component r_i of r is in $\mathfrak{p}_1$. By induction on m, it suffices to prove r_1 is in $\mathfrak{p}_1$.

Notice that

$$0 = re = r_1 e_1 + (\text{homogeneous terms of greater degree}).$$

So, $r_1 e_1$ is zero and we are done if $s = 1$, that is, if e is e_1. Notice that $r_1 e$ is $\sum_{i=2}^{s} r_1 e_i$ and has fewer nonzero homogeneous terms than e. Since s was minimal, the annihilator of $r_1 e$ properly contains $\mathfrak{p}_1$. Let r' be an element of the annihilator of $r_1 e$ that is not contained in $\mathfrak{p}_1$. We know $r' r_1$ is in $\mathfrak{p}_1$, r' is not in $\mathfrak{p}_1$, and $\mathfrak{p}_1$ is prime. Therefore, r_1 is in $\mathfrak{p}_1$. $\square$

Remark 1.4.2. We cannot delete the torsion-free assumption in the above theorem: Let R be any ring, let a and b be distinct elements of P such that $na = nb$ for some positive integer n, and let n_0 be the least such positive integer. Now $t^a - t^b$ is an element of $R[P]$ that is annihilated by $\sum_{m=0}^{n_0-1} t^{(n_0-m-1)a+mb}$. But, $t^a - t^b$ cannot be annihilated by any nonzero homogeneous element of $R[P]$ because P is cancellative.

1.5 Combinatorial $\mathbb{Z}[P]$-modules

Lemma 1.5.1. *let P be a finitely generated torsion-free monoid. If $\mathfrak{p}$ is a P-homogeneous prime of $\mathbb{Z}[P]$, then $\mathfrak{p} = (q) + \mathbb{Z}[K]$ where $(q) \subset \mathbb{Z}$ and $K \subset P$ are prime.*

Proof. Fix an arbitrary P-homogeneous prime $\mathfrak{p}$ of $\mathbb{Z}[P]$. The intersection $\mathfrak{p} \cap \mathbb{Z}$ must be prime. Let $(q) = \mathfrak{p} \cap \mathbb{Z}$. If $\mathfrak{p} = q\mathbb{Z}[P]$ we are done, take K to be the empty ideal. If $\mathfrak{p} \neq q\mathbb{Z}[P]$, pick an arbitrary P-homogeneous element nt^p of $\mathfrak{p} \setminus q\mathbb{Z}[P]$ with $n \in \mathbb{Z}$ and $p \in P$. In particular, $n \notin (q) = \mathfrak{p} \cap \mathbb{Z}$. So, $n \notin \mathfrak{p}$. Hence, $t^p \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. Let $K = \{p \in P \mid t^p \in \mathfrak{p}\}$ be the set of all such p. We have established $\mathfrak{p}$ is generated by q and $\mathbb{Z}[K]$. Now, it suffices to prove K is a prime ideal of P. If k is in K and p is in P, then $t^{p+k} = t^p \cdot t^k$ is in $\mathfrak{p}$ since $\mathfrak{p}$ is an ideal. So, K is an ideal. If K were not prime, there would exist p and p' in P such that $p + p'$ would be in K while neither p nor p' was in K. That is, there would exist t^p and $t^{p'}$ in $\mathbb{Z}[P]$ such that $t^p \cdot t^{p'}$ would be in $\mathfrak{p}$ while neither t^p nor t^q was in $\mathfrak{p}$ and $\mathfrak{p}$ would not be prime. Hence, K is prime. $\square$

Definition 1.5.2. Let P be a monoid. We say A is a *sub-P-set* of P^{gp} if A is a subset of P^{gp} and A is closed under the action of P on P^{gp} given by addition. That is, $P + A = A$. We say a sub-P-set of P^{gp} is a *fractional ideal* if there exists an element p of P such that $p + A$ is contained in P. A *combinatorial $\mathbb{Z}[P]$-module* is one isomorphic to $\mathbb{Z}[A, B] = \mathbb{Z}[A]/\mathbb{Z}[B]$, for some sub-$P$-sets A and B of P^{gp}. If K is an ideal of P and our module is also annihilated by $\mathbb{Z}[K]$, we will say it is a *combinatorial $\mathbb{Z}[P, K]$-module.*

If $\mathbb{Z}[A, B]$ is a combinatorial $\mathbb{Z}[P]$-module and E is any $\mathbb{Z}[P]$-module, we will write $\mathbb{Z}[A, B]E$ for $\mathbb{Z}[A]E/\mathbb{Z}[B]E$.

Corollary 1.5.3. *If P is a finitely generated torsion-free monoid and N is a finitely generated combinatorial $\mathbb{Z}[P]$-module, then N has a filtration*

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

with each $N_i/N_{i-1} \cong \mathbb{Z}[P, K]$ for some prime ideal $K \subset P$.

Proof. We may assume $N = \mathbb{Z}[A, B]$ where A and B are fractional ideals of P since P is a finitely generated torsion-free monoid. By Theorem 1.4.1, we know N has a filtration

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

with each $N_i/N_{i-1} \cong \mathbb{Z}[P]/\mathfrak{p}_i$ for some P-homogeneous prime ideal $\mathfrak{p}_i$. In particular, for $\mathfrak{p}_1$ we may take any associated prime of N. By induction on n, it suffices to prove that $\mathbb{Z}[A, B]$ has an associated prime of the form $\mathbb{Z}[K]$ for some prime $K \subset P$.

By Lemma 1.5.1, every associated prime of a finitely generated combinatorial $\mathbb{Z}[P]$-module is of the form $(q) + \mathbb{Z}[K]$ for some primes $K \subset P$ and $(q) \subset \mathbb{Z}$. Evidently, each combinatorial $\mathbb{Z}[P]$-module is a free abelian group and on any combinatorial $\mathbb{Z}[P]$-module every integer acts injectively. So, our prime is of the form $\mathbb{Z}[K]$ for some prime $K \subset P$. $\square$

Proposition 1.5.4. *If $P \to Q$ is an inclusion of finitely generated torsion-free monoids, K is an ideal of Q, $K' = K \cap P$, and N is a combinatorial $\mathbb{Z}[Q, K]$-module, then N is a direct sum of combinatorial $\mathbb{Z}[P, K']$-modules.*

Proof. Write $N = \mathbb{Z}[A, B]$ with A and B sub-Q-sets of Q^{gp} such that $A + K \subseteq B$. Pick a set C of coset representatives for Q^{gp}/P^{gp}. For each $c \in C$, let $A_c = \{q - c \mid q \in A \text{ and } q - c \in P^{gp}\}$ and let $B_c = \{q - c \mid q \in B \text{ and } q - c \in P^{gp}\}$. For each $c \in C$, A_c and B_c are sub-P-sets of P^{gp} such that $A_c + K' \subseteq B_c$. Therefore, $\mathbb{Z}[A_c, B_c]$ is a combinatorial $\mathbb{Z}[P, K']$-module. Furthermore, $\mathbb{Z}[A, B] \cong \bigoplus_{c \in C} \mathbb{Z}[A_c, B_c]$ as $\mathbb{Z}[P, K']$-modules. $\square$

1.6 Lorenzon's Algebra

Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a fine log scheme. For every local section $\overline{m}$ of $\overline{\mathcal{M}}_X$, the pre-image of $\overline{m}$ along the canonical map $\mathcal{M}_X \to \overline{\mathcal{M}}_X$ is an $\mathcal{O}_X^*$-torsor. So, the canonical map $\mathcal{M}_X \to \overline{\mathcal{M}}_X$ is an $\overline{\mathcal{M}}_X$-indexed family of $\mathcal{O}_X^*$-torsors. To each $\mathcal{O}_X^*$-torsor $\mathcal{E}$, we associate the contracted product $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$, where $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$ is the quotient of the product $\mathcal{E} \times \mathcal{O}_X$ by the equivalence relation $\sim$, where $(eu, f) \sim (e, uf)$ whenever e, u and f are respectively local sections of $\mathcal{E}$, $\mathcal{O}_X^*$ and $\mathcal{O}_X$. Each $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$ is an invertible sheaf. The association of $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$ to $\mathcal{E}$ and the $\overline{\mathcal{M}}_X$-indexed family of $\mathcal{O}_X^*$-torsors $\mathcal{M}_X \to \overline{\mathcal{M}}_X$ together yield an $\overline{\mathcal{M}}_X$-indexed family of invertible sheaves.

Furthermore, if $\overline{m}$ and $\overline{m}'$ are local sections $\overline{\mathcal{M}}_X$ with corresponding invertible sheaves $\mathcal{L}_{\overline{m}}$ and $\mathcal{L}_{\overline{m}'}$, then $\mathcal{L}_{\overline{m}+\overline{m}'}$ is isomorphic to $\mathcal{L}_{\overline{m}} \otimes \mathcal{L}_{\overline{m}'}$ (see Lorenzon [25]). This family of invertible sheaves is a ring object in the comma category of sheaves of sets on X over $\overline{\mathcal{M}}_X$, $\mathrm{Sh}(X)/\overline{\mathcal{M}}_X$, and the part indexed by the identity section of $\overline{\mathcal{M}}_X$ is $\mathcal{O}_X$. Lorenzon calls this $\overline{\mathcal{M}}_X$-indexed $\mathcal{O}_X$-algebra the canonical algebra of the log scheme. Let the algebra $\mathcal{A}_X$ be the direct sum over sections of $\overline{\mathcal{M}}_X$ of these invertible sheaves with multiplication given by tensor product over $\mathcal{O}_X$ extended by $\mathcal{O}_X$-linearity. Lorenzon calls this algebra the algebra induced on X by the canonical algebra of the log scheme. The canonical algebra consists of the homogeneous pieces of our $\overline{\mathcal{M}}_X$-graded algebra.

Warning 1.6.1. Lorenzon uses $(\mathcal{A}_X)_*$ to denote the algebra we denote by $\mathcal{A}_X$ and he uses $\mathcal{A}_X$ to denote the canonical algebra of the log scheme.

It happens that $\mathcal{A}_X$ is the $\mathcal{O}_X$-algebra $\mathcal{O}_X[\mathcal{M}_X]/(t^m - \alpha(m))_{m \in \mathcal{M}_X^*}$. Given an ideal sheaf $\mathcal{K}$ of $\mathcal{M}_X$, let $\mathcal{I}(\mathcal{K})$ be the image in $\mathcal{A}_X$ of the ideal sheaf $\mathcal{O}_X[\mathcal{K}]$ of $\mathcal{O}_X[\mathcal{M}_X]$ along the canonical homomorphism $\mathcal{O}_X[\mathcal{M}_X] \to \mathcal{A}_X$.

Example 1.6.2. ($\mathbf{P}^1$ with a marked point) Let k be a field. Consider the point $p = (x)$ on $\mathbf{P}^1 = \operatorname{Proj} k[x, y]$. Let $\mathcal{M}_{\mathbf{P}^1}$ be the log structure that is trivial on the open subset $\mathbf{P}^1 \setminus \{p\}$ and induced by the $\mathbb{N} \to k[\frac{x}{y}]$, $1 \mapsto \frac{x}{y}$ on the affine open subscheme $U = \operatorname{Spec} k[\frac{x}{y}]$ containing p. Here $\overline{\mathcal{M}}_{\mathbf{P}^1}$ is the skyscraper sheaf with stalk $\mathbb{N}$ at p and the $\mathcal{O}^*_{\mathbf{P}^1}$-torsor associated to the natural number n is generated by $(\frac{x}{y})^n$ on U and by 1 on $\mathbf{P}^1 \setminus \{p\}$. So, the invertible sheaf associated to the natural number n has local basis $(\frac{x}{y})^n$ on U and 1 on $\mathbf{P}^1 \setminus \{p\}$. That is, the invertible sheaf associated to the natural number n is $\mathcal{L}(-p)^{\otimes n}$ and $\mathcal{A}_{\mathbf{P}^1}$ is the $\mathcal{O}_{\mathbf{P}^1}$-algebra $\bigoplus_{n \in \mathbb{N}} \mathcal{L}(-p)^{\otimes n}$.

Proposition 1.6.3. *Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X) = \operatorname{Spec}(P \xrightarrow{\beta} R)$ be a fine log scheme and let $\beta : P \to R$ be local at $x = \mathfrak{p}$, then*

$$\mathcal{A}_{X,x} \cong \mathcal{O}_{X,x}[P]/(t^p - \beta(p))_{p \in P^*}.$$

Proof. To prove this, we write $\mathcal{M}_{X,x} \cong \mathcal{O}^*_{X,x} \oplus P/\{\beta(u)^{-1} \oplus u \mid u \in P^*\}$ and interchange the order in which quotients are taken. Instead of first applying the congruence that forms $\mathcal{M}_{X,x}$ from $\mathcal{O}^*_{X,x} \oplus P$, we first identify the copy of $\mathcal{O}^*_{X,x}$ in $\mathcal{O}^*_{X,x} \oplus P$ with the copy of $\mathcal{O}^*_{X,x}$

in $\mathcal{O}_{X,x}$:

$$\begin{aligned}\mathcal{A}_{X,x} &= \mathcal{O}_{X,x}[\mathcal{M}_{X,x}]/(t^m - \alpha(m))_{m\in\mathcal{M}^*_{X,x}} \\ &= \mathcal{O}_{X,x}[\mathcal{O}^*_{X,x} \oplus P]/(t^{(1,p)} - t^{(\beta(p),0)}, t^{(u,0)} - u)_{p\in P^*, u\in\mathcal{O}^*_{X,x}} \\ &= \mathcal{O}_{X,x}[P]/(t^p - \beta(p))_{p\in P^*}\end{aligned}$$

□

In particular, $\mathcal{A}_{X,x} \cong \mathcal{O}_{X,x}[\overline{\mathcal{M}}_{X,x}]$ when $\overline{\mathcal{M}}_{X,x}$ is torsion-free by Proposition 1.3.3.

Chapter 2

t-Flatness

2.1 First Properties of t-Flatness

Definition 2.1.1. Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be an fine log scheme, let x be a point of X, let $\mathcal{K}$ be an ideal of $\mathcal{M}_X$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules such that $\mathcal{I}(\mathcal{K})_x \mathcal{F}_x = 0$. We say $\mathcal{F}$ is *$\mathcal{M}_X$-flat relative to $\mathcal{K}$ at x* if for all ideals $\mathcal{J}$ of $\mathcal{M}_X$ containing $\mathcal{K}$ we have

$$\mathrm{Tor}_1^{\mathcal{A}_{X,x}/\mathcal{I}(\mathcal{K})_x}(\mathcal{A}_{X,x}/\mathcal{I}(\mathcal{J})_x, \mathcal{F}_x) = 0$$

We say $\mathcal{F}$ is $\mathcal{M}_X$-flat at x if $\mathcal{F}$ is $\mathcal{M}_X$-flat relative to the constant ideal sheaf with empty stalks at x.

Definition 2.1.2. Let P be a finitely generated monoid, let $K \subseteq P$ be an ideal, and let E be a nonzero $\mathbb{Z}[P, K]$-module. We say E has *t-flat dimension d relative to K* if

$$d = \sup\{i \mid \mathrm{Tor}_i^{\mathbb{Z}[P,K]}(\mathbb{Z}[P, J], E) \neq 0,\ J \text{ an ideal of } P \text{ containing } K\}.$$

If $E = 0$, we say E has t-flat dimension 0 relative to K. We say E is *t-flat relative to K* if

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P, J], E) = 0$$

for all ideals $J \subseteq P$ containing K. If E is t-flat relative to $\emptyset$, we simply say E is t-flat.

Later, Theorem 2.2.2, we will prove E is t-flat relative to K if and only if E has t-flat dimension 0 relative to K.

Proposition 2.1.3. *Let P be a finitely generated torsion-free monoid, let R be a Noetherian ring, let $\mathfrak{p} \subset R$ be prime, let $\beta : P \to R$ be a monoid homomorphism with respect to*

multiplication on R, let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X) = \operatorname{Spec}(P \xrightarrow{\beta} R)$, and let x be the point on X corresponding to $\mathfrak{p}$. Suppose $\overline{\mathcal{M}}_{X,x}$ is torsion-free and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We consider $\mathcal{F}_x$ to be a $\mathbb{Z}[P]$-module along the map $\mathbb{Z}[P] \to \mathcal{O}_{X,x}$ induced by β. If $K \subseteq P$ is an ideal such that $\mathbb{Z}[K]$ annihilates $\mathcal{F}_x$, then

$$\operatorname{Tor}_i^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J], \mathcal{F}_x) \cong \operatorname{Tor}_i^{\mathcal{A}_{X,x}/\mathcal{I}(K)_x}(\mathcal{A}_{X,x}/\mathcal{I}(J)_x, \mathcal{F}_x), \forall i \geq 0$$

for all ideals J of P containing K.

Proof. Since both of these modules are $\mathcal{O}_{X,x}$- modules, both sides are zero if $J \not\subseteq \beta^{-1}(\mathfrak{p})$ and we may assume $R = \mathcal{O}_{X,x}$. By Proposition 1.3.3, we may assume $P \cong \overline{\mathcal{M}}_{X,x}$. In particular, we may assume $\mathcal{A}_{X,x} = R[P]$ by Proposition 1.6.3. Fix ideals $K \subseteq J \subseteq P$. Let

$$\mathbf{F}_\bullet : \qquad \cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z}[P,J] \to 0$$

be a P^{gp}-graded free resolution of the $\mathbb{Z}[P,K]$-module $\mathbb{Z}[P,J]$. Since $\mathbb{Z}[P,J]$ is a free $\mathbb{Z}$-module,

$$\mathbf{F}'_\bullet : \qquad \cdots \to F_2 \otimes_{\mathbb{Z}} R \to F_1 \otimes_{\mathbb{Z}} R \to F_0 \otimes_{\mathbb{Z}} R \to R[P,J] \to 0$$

is a P^{gp}-graded free resolution of $R[P,J]$ as a $R[P,K]$-module. Furthermore,

$$\begin{aligned} F_i \otimes_{\mathbb{Z}[P,K]} \mathcal{F}_x &\cong F_i \otimes_{\mathbb{Z}[P,K]} R[P,K] \otimes_{R[P,K]} \mathcal{F}_x \\ &\cong F_i \otimes_{\mathbb{Z}[P,K]} \mathbb{Z}[P,K] \otimes_{\mathbb{Z}} R \otimes_{R[P,K]} \mathcal{F}_x \\ &\cong F_i \otimes_{\mathbb{Z}} R \otimes_{R[P,K]} \mathcal{F}_x \end{aligned}$$

Therefore, $\operatorname{Tor}_i^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J], \mathcal{F}_x)$, the ith cohomology module of

$$\mathbf{F}_\bullet \otimes_{\mathbb{Z}[P,K]} \mathcal{F}_x,$$

and $\operatorname{Tor}_i^{R[P,K]}(R[P,J], \mathcal{F}_x)$, the ith cohomology module of

$$\mathbf{F}'_\bullet \otimes_{R[P,K]} \mathcal{F}_x,$$

are isomorphic for all $i \geq 0$. □

Corollary 2.1.4. *Let P be a finitely generated torsion-free monoid, let R be a Noetherian ring, let $\mathfrak{p} \subset R$ be prime, let $\beta : P \to R$ be a monoid homomorphism with respect to multiplication on R, let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X) = \operatorname{Spec}(P \xrightarrow{\beta} R)$, and let x be the point on X*

corresponding to $\mathfrak{p}$. *Suppose* $\overline{\mathcal{M}}_{X,x}$ *is torsion-free and let* $\mathcal{F}$ *be an* $\mathcal{O}_X$*-module. We consider* $\mathcal{F}_x$ *to be a* $\mathbb{Z}[P]$*-module along the map* $\mathbb{Z}[P] \to \mathcal{O}_{X,x}$ *induced by* β. *If* $K \subseteq P$ *is an ideal such that* $\mathbb{Z}[K]$ *annihilates* $\mathcal{F}_x$ *and* $\mathcal{K} \subseteq \mathcal{M}_X$ *is the ideal generated by the image of* K, *then* $\mathcal{F}$ *is* $\mathcal{M}_X$*-flat relative to* $\mathcal{K}$ *at* $x \in X$ *if and only if* $\mathcal{F}_x$ *is t-flat relative to* K.

2.2 Local Criterion for t-Flatness

Let I be an ideal of a ring B. Recall that a B-module E is said to be I-adically ideal-separated if $\mathfrak{a} \otimes E$ is separated for the I-adic topology for every finitely generated ideal $\mathfrak{a}$ of B. In particular, if R is a Noetherian B-algebra, $IR \subseteq \mathrm{rad}(R)$, and E is a finitely generated R-module, then E is I-adically ideal-separated.

Theorem 2.2.1. *(Local Criterion for Flatness; Matsumura [26, Theorem 22.3]) Let* I *be an ideal of a ring* B *and let* E *be a* B*-module. Set* $B_n = B/I^{n+1}$ *for each integer* $n \geq 0$, $E_n = E/I^{n+1}E$ *for each integer* $n \geq 0$, $\mathrm{gr}(B) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, *and* $\mathrm{gr}(E) = \bigoplus_{n\geq 0} I^nE/I^{n+1}E$. *Let*

$$\mu_n : (I^n/I^{n+1}) \otimes_{B_0} E_0 \to I^nE/I^{n+1}E$$

be the standard map for each $n \geq 0$, *and let*

$$\mu : \mathrm{gr}(B) \otimes_{B_0} E_0 \to \mathrm{gr}(E)$$

be the direct sum of the γ_n. *Suppose (a)* I *is nilpotent or (b)* B *is Noetherian and* E *is* I*-adically ideal-separated. Then the following are equivalent,*

(1) E *is flat over* B;

(2) $\mathrm{Tor}_1^B(N, E) = 0$ *for every* B_0*-module* N;

(3) E_0 *is flat over* B_0 *and the canonical surjection*

$$I \otimes_B E \to IE$$

is an isomorphism;

(3') E_0 *is flat over* B_0 *and* $\mathrm{Tor}_1^B(B_0, E) = 0$;

(4) E_0 *is flat over* B_0 *and* μ_n *is an isomorphism for every* $n \geq 0$;

(4') E_0 is flat over B_0 and μ is an isomorphism;

(5) E_n is flat over B_n for every $n \geq 0$.

In fact, the implications (1)⇒(2)⇔(3)⇔(3')⇒(4)⇒(5) hold without the assumptions (a) and (b).

We will model the proof of our Local Criterion for t-Flatness on the proof of this theorem. Let P be a finitely generated torsion-free monoid. We consider the case where $B = \mathbb{Z}[P, K]$, $I = \mathbb{Z}[P^+, K]$ and one of the following two conditions holds: R is a Noetherian B-algebra, $IR \subseteq \mathrm{rad}(R)$, and E is a finitely generated R-module or I is nilpotent.

Let d be the dimension of $\overline{P^{sat}}$ and let $\varphi : \overline{P^{sat}} \to \mathbb{N}^d$ be an inclusion as in Proposition 1.2.7. Pick d $\mathbb{Q}$-linearly independent positive real numbers $\{\gamma_1, \gamma_2, \ldots, \gamma_d\}$, let the monoid homomorphism $\psi : \mathbb{N}^d \to \mathbb{R}$ be given by $(n_1, n_2, \ldots, n_d) \mapsto \sum_{i=1}^{d} n_i\gamma_i$, let the monoid homomorphism $\nu : P \to \mathbb{R}$ be the composition $P \xrightarrow{canonical} \overline{P^{sat}} \xrightarrow{\varphi} \mathbb{N}^d \xrightarrow{\psi} \mathbb{R}$, and let Γ be the image of ν. We order Γ with the order induced by the standard ordering of $\mathbb{R}$. Notice that Γ is well ordered.

Set$I_\gamma = (t^p)_{\gamma \leq \nu(p),\, p \in P \setminus K}$ for each $\gamma \in \Gamma$, $K_\gamma^+ = \{p \in P \mid \nu(p) > \gamma\}$ for each $\gamma \in \Gamma$, $I_\gamma^+ = I_{\min\{\gamma' \mid \gamma < \gamma'\}}$ for each $\gamma \in \Gamma$, $B_\gamma = B/I_\gamma^+$ for each $\gamma \in \Gamma$, $E_\gamma = E/I_\gamma^+ E$ for each $\gamma \in \Gamma$, let $\mathrm{gr}_\gamma(B) = I_\gamma / I_\gamma^+$ for each $\gamma \in \Gamma$, let $\mathrm{gr}_\gamma(E) = I_\gamma E / I_\gamma^+ E$ for each $\gamma \in \Gamma$, let $\mathrm{gr}(B)$ be the associated graded ring of the filtration $\{I_\gamma \mid \gamma \in \Gamma\}$, let $\mathrm{gr}(E)$ be the associated graded module of the filtration $\{I_\gamma E \mid \gamma \in \Gamma\}$, let

$$\mu_\gamma : \mathrm{gr}_\gamma(B) \otimes_{B_0} E_0 \to \mathrm{gr}_\gamma(E)$$

be the multiplication map for each $n \geq 0$, and let

$$\mu : \mathrm{gr}(B) \otimes_{B_0} E_0 \to \mathrm{gr}(E)$$

be the direct sum of the μ_γ.

Theorem 2.2.2. *(Local Criterion for t-Flatness) Continuing the notation above, $\mathrm{gr}(B) \cong B$ and if R is a Noetherian B-algebra, $IR \subseteq \mathrm{rad}(R)$, and E is a finitely generated R-module or if I is nilpotent, then the following are equivalent:*

(1) E is t-flat relative to K.

(2) $\mathrm{Tor}_i^B(N, E) = 0$ for all $i > 0$ and every combinatorial B-module N.

(3) The canonical surjection

$$I \otimes_B E \to IE$$

is an isomorphism.

(3') $\mathrm{Tor}_1^B(B_0, E) = 0$.

(4) μ_γ *is an isomorphism for all* $\gamma \in \Gamma$.

(4') μ *is an isomorphism.*

(5) E_γ *is t-flat relative to* $K_\gamma^+ \cup K$ *for all* $\gamma \in \Gamma$.

(6) The multiplication map

$$(I^n/I^{n+1}) \otimes_{B_0} E_0 \to I^n E/I^{n+1}E$$

is an isomorphism for all $n \in \mathbb{N}$.

(7) $E/I^{n+1}E$ *is t-flat relative to* $(n+1)P^+ \cup K$ *for all* $n \in \mathbb{N}$.

In fact, (1)⇔(2)⇒(3)⇔(3')⇔(4)⇔(4')⇔(5)⇔(6)⇔(7) without any extra assumptions on I *or* E.

Proof. First, consider the underlying group of $\mathrm{gr}(B)$. Since each ideal I_γ is P-homogeneous,

$$\mathrm{gr}(B) = \bigoplus_{\gamma \in \Gamma} I_\gamma/I_\gamma^+ \cong \bigoplus_{\gamma\in\Gamma}\left(\bigoplus_{\substack{\nu(p)=\gamma \\ p \in P\setminus K}} \mathbb{Z}t^p\right) \cong \bigoplus_{p \in P\setminus K} \mathbb{Z}t^p \cong B.$$

Furthermore, multiplication is given by

$$t^p \cdot t^q = \begin{cases} t^{p+q} & \text{if } p+q \notin K, \\ 0 & \text{otherwise,} \end{cases}$$

in $\mathrm{gr}(B)$ since $\nu(p+q) = \nu(p) + \nu(q)$. Since $\{t^p \mid p \in P \setminus K\}$ is a $\mathbb{Z}$-basis for the free group $\mathrm{gr}(B)$, $\mathrm{gr}(B) \cong B$ as rings as well.

(2)⇒(1): If $J \subset P$ is an arbitrary ideal containing K, then $\mathbb{Z}[P, J]$ is a combinatorial B-module. Hence, by (2), $\mathrm{Tor}_1^B(\mathbb{Z}[P, J], E) = 0$ for every ideal $J \subset P$ containing K. That is, E is t-flat relative to K.

(1)⇒(2): First, we will treat the finitely generated combinatorial B-modules. We proceed by induction on i. Let $i = 1$. If N is a finitely generated combinatorial B-module, then N has a filtration

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

with each $N_l/N_{l-1} \cong \mathbb{Z}[P, J_l]$ for some prime ideal $J_l \subset P$ according to Corollary 1.5.3. Since each N_l is a submodule of N, each N_l is annihilated by $I(K)$. Furthermore, each $\mathbb{Z}[P, J_l] \cong N_l/N_{l-1}$. So, each $\mathbb{Z}[P, J_l]$ is a quotient of a module annihilated by $I(K)$. Hence, each $\mathbb{Z}[P, J_l]$ is annihilated by $I(K)$ as well. Therefore, each J_l contains K. Now we proceed by induction on the length of our filtration. If $n = 1$, then $N = N_1 \cong \mathbb{Z}[P, J_1]$ and we are done. If $n > 1$, assume $\mathrm{Tor}_1^B(N', E) = 0$ for every combinatorial B-module N' whose filtration has length $n - 1$. We have

$$0 \to N_{n-1} \to N \to \mathbb{Z}[P, J_n] \to 0.$$

Once we take the tensor product with E, we obtain the long exact sequence

$$\cdots \to \mathrm{Tor}_1^B(N_{n-1}, E) \to \mathrm{Tor}_1^B(N, E) \to \mathrm{Tor}_1^B(\mathbb{Z}[P, J_n], E) \to \cdots$$

Since E is t-flat relative to K, $\mathrm{Tor}_1^B(\mathbb{Z}[P, J_n], E) = 0$. Furthermore,

$$\mathrm{Tor}_1^B(N_{n-1}, E) = 0$$

by the induction hypothesis. So, we have $\mathrm{Tor}_1^B(N, E) = 0$ for every finitely generated combinatorial B-module N. Now suppose $i > 1$ and

$$\mathrm{Tor}_{i-1}^B(N', E) = 0$$

for every finitely generated combinatorial B-module N'. Let $J \subseteq P$ be an ideal containing K. We have

$$0 \to \mathbb{Z}[J, K] \to B \to \mathbb{Z}[P, J] \to 0.$$

Once we take the tensor product with E, we obtain the long exact sequence

$$\cdots \to 0 \to \mathrm{Tor}_i^B(\mathbb{Z}[P, J], E) \to \mathrm{Tor}_{i-1}^B(\mathbb{Z}[J, K], E) \to 0 \to \cdots$$

Since $\mathrm{Tor}_{i-1}^B(\mathbb{Z}[J, K], E) = 0$ by the induction hypothesis,

$$\mathrm{Tor}_i^B(\mathbb{Z}[P, J], E) = 0.$$

Any combinatorial B-module is the union of its finitely generated combinatorial submodules. Since right exact functors commute with colimits and filtered colimits are exact in module categories, we are done.

(3)⇔(3'): Notice that $\mathrm{Tor}_1^B(B_0, E)$ is the kernel of the surjective map in (3).

(1)⇒(3'): (1) implies (3') follows immediately from the definition of t-flatness.

(3')⇒(1): Let $S = \{J \mid \mathrm{Tor}_1^B(\mathbb{Z}[P,J], E) \neq 0\}$. If S is nonempty, then S has a maximal element J since the ideals of P satisfy the ascending chain condition. By Lemma 2.2.3, J is prime. Let $q \in P^+ \setminus J$. Since J is prime, we have a short exact sequence

$$0 \to \mathbb{Z}[P,J] \xrightarrow{\cdot t^q} \mathbb{Z}[P,J] \to \mathbb{Z}[P, J \cup (P+q)] \to 0.$$

This sequence yields the long exact Tor sequence

$$\cdots \to \mathrm{Tor}_1^B(\mathbb{Z}[P,J],E) \xrightarrow{\cdot t^q} \mathrm{Tor}_1^B(\mathbb{Z}[P,J],E) \to 0$$

Now apply Nakayama's Lemma. We conclude that

$$\mathrm{Tor}_1^B(\mathbb{Z}[P,J],E)_{\mathfrak{p}} = 0$$

at every prime $\mathfrak{p}$ containing an element of $P^+ \setminus J$. If I is nilpotent, then every prime contains every element of $P^+ \setminus J$. If E is a finitely generated module over some B-algebra R such that $IR \subseteq \mathrm{rad}(R)$, every maximal ideal in the support of E contains every element of $P^+ \setminus J$. In either case,

$$\mathrm{Tor}_1^B(\mathbb{Z}[P,J],E)_{\mathfrak{m}} = 0$$

at every maximal ideal $\mathfrak{m} \subset B$ in the support of E. Therefore,

$$\mathrm{Tor}_1^B(\mathbb{Z}[P,J],E) = 0$$

and E is t-flat relative to K.

(2)⇒(4): By (2) we have $\mathrm{Tor}_1^B(\mathrm{gr}_\gamma(B),E) = 0$,

$$\mathrm{Tor}_1^B(B_\gamma, E) = 0$$

and $\mathrm{Tor}_1^B(B/I_\gamma, E) = 0$ for all $\gamma \in \Gamma$. Considering

$$0 \to I_\gamma^+ \to I_\gamma \to \mathrm{gr}_\gamma(B) \to 0,$$

we see that

$$0 \to I_\gamma^+ \otimes_B E \to I_\gamma \otimes_B E \to \mathrm{gr}_\gamma(B) \otimes_B E \to 0$$

is exact. By the same sort of argument used to establish (3)⇔(3'),

$$I_\gamma^+ \otimes_B E \cong I_\gamma^+ E \text{ and } I_\gamma \otimes_B E \cong I_\gamma E.$$

So, $\mathrm{gr}_\gamma(B) \otimes_B E \cong \mathrm{gr}_\gamma(E)$.

(4)⇔(4'): Evident.

(4)⇒(5): Fix $\gamma \in \Gamma$. For each $\gamma' \leq \gamma$, consider the commutative diagram

$$\begin{array}{ccccccccc} & & (I_{\gamma'}^+/I_\gamma^+) \otimes_{B_0} E_0 & \to & (I_{\gamma'}/I_\gamma^+) \otimes_{B_0} E_0 & \to & \mathrm{gr}_{\gamma'}(B) \otimes_{B_0} E_0 & \to & 0 \\ & & \alpha_{\gamma'}^+ \downarrow & & \alpha_{\gamma'} \downarrow & & \mu_{\gamma'} \downarrow & & \\ 0 & \to & I_{\gamma'}^+ E/I_\gamma^+ E & \to & I_{\gamma'} E/I_\gamma^+ E & \to & \mathrm{gr}_{\gamma'}(E) & \to & 0 \end{array}$$

with exact rows. Notice that there are only finitely many elements of Γ less than γ. We will prove α_0^+ is an isomorphism by decreasing induction on γ'. When $\gamma' = \gamma$, $\alpha_{\gamma'}^+ = \alpha_\gamma^+ : 0 \to 0$ is an isomorphism. By assumption, each $\mu_{\gamma'}$ is an isomorphism. So, whenever $\alpha_{\gamma'}^+$ is an isomorphism, $\alpha_{\gamma'}$ is also an isomorphism. Therefore, each $\alpha_{\gamma'}$ is an isomorphism. In particular, α_0^+ is an isomorphism. That is, if we replace K with $K' = K_\gamma^+ \cup K$ and E by E_γ, we have

$$\mathbb{Z}[P^+, K'] \otimes_{\mathbb{Z}[P,K']} E_\gamma \cong \mathbb{Z}[P^+, K'] E_\gamma.$$

So, since (1) is equivalent to (3), we have (5).

(5)⇒(1): If I is nilpotent, $E_\gamma = E$ and $K_\gamma^+ \cup K = K$ for γ sufficiently large, and we are done. It suffices to prove that the standard map $\varphi : \mathbb{Z}[J,K] \otimes E \to E$ is injective for any log ideal $\mathbb{Z}[J,K]$ of B. Let $\mathfrak{a} = \mathbb{Z}[J,K]$. I is finitely generated. Fix a finite generating set X for I, let $\gamma = \max\{\nu(x) \mid x \in X\}$, and let $\gamma' = \min\{\nu(x) \mid x \in X\}$. For all n, $I_{n\gamma} \subseteq I^n \subseteq I_{n\gamma'}$. Since E is I-adically ideal-separated, we know $\bigcap_{\gamma \in \Gamma} I_\gamma(\mathfrak{a} \otimes_B E) = 0$. Therefore, it suffices to prove $\ker(\varphi) \subseteq I_\gamma(\mathfrak{a} \otimes_B E)$ for all $\gamma \in \Gamma$. The Artin-Rees lemma tells us that, for a fixed n and for sufficiently large $k > n$, $I^k \cap \mathfrak{a} \subseteq I^n \mathfrak{a}$. So, for a fixed n and for sufficiently large $k > n$, $I_{k\gamma} \cap \mathfrak{a} \subseteq I_{n\gamma'} \mathfrak{a}$. Now consider the natural map

$$\mathfrak{a} \otimes_B E \xrightarrow{f} (\mathfrak{a}/I_{k\gamma} \cap \mathfrak{a}) \otimes_B E \xrightarrow{g} (\mathfrak{a}/I_{n\gamma'}\mathfrak{a}) \otimes_B E = (\mathfrak{a} \otimes_B E)/I_{n\gamma'}(\mathfrak{a} \otimes_B E)$$

Since the $\mathbb{Z}[P, I_{k\gamma} \cup K]$-module $E/I_{k\gamma}E$ is t-flat relative to $I_{k\gamma} \cup K$, the map

$$(\mathfrak{a}/I_{k\gamma} \cap \mathfrak{a}) \otimes_B E = (\mathfrak{a}/I_{k\gamma} \cap \mathfrak{a}) \otimes_{\mathbb{Z}[P, I_{k\gamma} \cup K]} E/I_{k\gamma}E \to E/I_{k\gamma}E$$

is injective, so that from the commutative diagram

$$\begin{array}{ccc} \mathfrak{a}\otimes_B E & \xrightarrow{f} & (\mathfrak{a}/I_{k\gamma}\cap\mathfrak{a})\otimes_B E \\ \varphi\downarrow & & \downarrow \\ E & \to & E/I_{k\gamma}E \end{array}$$

we get $\ker(\varphi)\subseteq\ker(f)\subseteq\ker(gf)=I_{n\gamma'}(\mathfrak{a}\otimes_B E)$. This is what we wanted to prove.

(3')⇔(4)⇔(5): Fix γ and apply the previous arguments with K replaced by $K'=K_\gamma^+\cup K$ and E replaced by $E/K'E$. In this case, $\mathbb{Z}[P^+,K']$ is nilpotent. So, (4) and (5) are both equivalent to

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(B_0,E/K'E)=0.$$

But, $\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(B_0,E/K'E)$ is the kernel of the canonical map

$$\varphi:(I/I_\gamma^+)\otimes_{B_\gamma}E_\gamma\to E_\gamma$$

and $(I/I_\gamma^+)\otimes_{B_\gamma}E_\gamma\cong(I/I_\gamma^+)\otimes_{B_\gamma}B_\gamma\otimes_B E\cong(I/I_\gamma^+)\otimes_B E$. So, $\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(B_0,E/K'E)$ is isomorphic to $\mathrm{Tor}_1^B(B_0,E)$ the kernel of $(I/I_\gamma^+)\otimes_B E\to E_\gamma$.

(2)⇒(6): By (2) we have $\mathrm{Tor}_1^B(I^n/I^{n+1},E)=0$,

$$\mathrm{Tor}_1^B(B/I^{n+1},E)=0$$

and $\mathrm{Tor}_1^B(B/I^n,E)=0$ for all $n\in\mathbb{N}$. Considering

$$0\to I^{n+1}\to I^n\to I^n/I^{n+1}\to 0,$$

we see that

$$0\to I^{n+1}\otimes_B E\to I^n\otimes_B E\to(I^n/I^{n+1})\otimes_B E\to 0$$

is exact. By the same sort of argument used to establish (3)⇔(3'),

$$I^{n+1}\otimes_B E\cong I^{n+1}E \text{ and } I^n\otimes_B E\cong I^nE.$$

So, $(I^n/I^{n+1})\otimes_B E\cong I^nE/I^{n+1}E$.

(6)⇒(7): Fix a positive integer n. For each $i\le n$, consider the commutative diagram

$$\begin{array}{ccccccccc} & & (I^{i+1}/I^{n+1})\otimes_{B_0}E_0 & \to & (I^i/I^{n+1})\otimes_{B_0}E_0 & \to & (I^i/I^{i+1})\otimes_{B_0}E_0 & \to & 0 \\ & & \alpha_{i+1}\downarrow & & \alpha_i\downarrow & & \beta_i\downarrow & & \\ 0 & \to & I^{i+1}E/I^{n+1}E & \to & I^iE/I^{n+1}E & \to & I^iE/I^{i+1}E & \to & 0 \end{array}$$

with exact rows. We will prove α_1 is an isomorphism by decreasing induction on i. When $i = n$, $\alpha_{i+1} = \alpha_{n+1} : 0 \to 0$ is an isomorphism. By assumption, each β_i is an isomorphism. So, whenever α_{i+1} is an isomorphism, α_i is also an isomorphism. Therefore, each α_i is an isomorphism. In particular, α_1 is an isomorphism. That is, we have

$$(I/I^{n+1}) \otimes_{B/I^{n+1}} E/I^{n+1}E \cong IE/I^{n+1}E.$$

So, since (1) is equivalent to (3), we have (7).

(7)⇒(1): If I is nilpotent, $E/I^{n+1} = E$ and $(n+1)P^+ \cup K = K$ for n sufficiently large, and we are done. It suffices to prove that the standard map $\varphi : \mathbb{Z}[J, K] \otimes E \to E$ is injective for any log ideal $\mathbb{Z}[J, K]$ of B. Let $\mathfrak{a} = \mathbb{Z}[J, K]$. I is finitely generated. Since E is I-adically ideal-separated, we know $\bigcap_{n \in \mathbb{N}} I^n(\mathfrak{a} \otimes_B E) = 0$. Therefore, it suffices to prove $\ker(\varphi) \subseteq I^n(\mathfrak{a} \otimes_B E)$ for all $n \in \mathbb{N}$. The Artin-Rees lemma tells us that, for a fixed n and for sufficiently large $k > n$, $I^k \cap \mathfrak{a} \subseteq I^n\mathfrak{a}$. Now consider the natural map

$$\mathfrak{a} \otimes_B E \xrightarrow{f} (\mathfrak{a}/I^k \cap \mathfrak{a}) \otimes_B E \xrightarrow{g} (\mathfrak{a}/I^n\mathfrak{a}) \otimes_B E = (\mathfrak{a} \otimes_B E)/I^n(\mathfrak{a} \otimes_B E)$$

Since the $\mathbb{Z}[P, kP^+ \cup K]$-module E/I^kE is t-flat relative to $kP^+ \cup K$, the map

$$(\mathfrak{a}/I^k \cap \mathfrak{a}) \otimes_B E = (\mathfrak{a}/I^k \cap \mathfrak{a}) \otimes_{\mathbb{Z}[P,kP^+ \cup K]} E/I^kE \to E/I^kE$$

is injective, so that from the commutative diagram

$$\begin{array}{ccc}
\mathfrak{a} \otimes_B E & \xrightarrow{f} & (\mathfrak{a}/I^k \cap \mathfrak{a}) \otimes_B E \\
{\scriptstyle\varphi}\downarrow & & \downarrow \\
E & \to & E/I^kE
\end{array}$$

we get $\ker(\varphi) \subseteq \ker(f) \subseteq \ker(gf) = I^n(\mathfrak{a} \otimes_B E)$. This is what we wanted to prove.

(3')⇔(6)⇔(7): Fix n and apply the previous arguments with K replaced by $(n+1)P^+ \cup K$ and E replaced by $E/I^{n+1}E$. In this case, I/I^{n+1} is nilpotent. So, (6) and (7) are both equivalent to

$$\mathrm{Tor}_1^{B/I^{n+1}}(B_0, E/I^{n+1}E) = 0.$$

But, $\mathrm{Tor}_1^{B/I^{n+1}}(B_0, E/I^{n+1}E)$ is the kernel of the canonical map

$$\varphi : (I/I^{n+1}) \otimes_{B/I^{n+1}} (E/I^{n+1}E) \to E/I^{n+1}E$$

and $(I/I^{n+1}) \otimes_{B/I^{n+1}} (E/I^{n+1}E) \cong (I/I^{n+1}) \otimes_{B/I^{n+1}} B/I^{n+1} \otimes_B E \cong (I/I^{n+1}) \otimes_B E$. So, $\mathrm{Tor}_1^{B/I^{n+1}}(B_0, E/I^{n+1}E)$ is isomorphic to $\mathrm{Tor}_1^B(B_0, E)$ the kernel of $(I/I^{n+1}) \otimes_B E \to E/I^{n+1}E$. □

Lemma 2.2.3. *Continuing the above notation, if J' is an ideal of P containing K such that*

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J'],E) \neq 0,$$

then there exists a prime J containing J' such that

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J],E) \neq 0.$$

Proof. Let $S = \{J'' \mid \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J''],E) \neq 0\}$. Since S is nonempty, S has a maximal element J since the ideals of P satisfy the ascending chain condition. Suppose J is an ideal of P containing J' such that

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J],E) \neq 0$$

and no ideal properly containing J has this property. We will prove J is prime by contradiction. Suppose J were not prime. Using Theorem 1.4.1, write

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = \mathbb{Z}[P,J]$$

such that each $N_{i+1}/N_i \cong \mathbb{Z}[P,J_i]$ for some prime ideal $J_i \in P$. As before, each J_i contains J. Since J is assumed to not be prime, these containments are proper. We get a series of short exact sequences

$$0 \to N_i \to N_{i+1} \to \mathbb{Z}[P,J_i] \to 0$$

Take the various long exact Tor sequences to get

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(N_i,E) \to \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(N_{i+1},E) \to 0$$

by the maximality of J. That is

$$\begin{aligned}
\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J],E) &= \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(N_n,E) \\
&\cong \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(N_1,E) \\
&\cong \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J_0],E) \\
&= 0
\end{aligned}$$

This contradicts the choice of J. So, J must be prime. □

In particular, a $\mathbb{Z}[P,K]$-module E is t-flat (relative to K) if and only if $E_{\mathfrak{p}}$ is flat over $\mathbb{Z}[P,K]_{\mathfrak{p}}$ for every log prime ideal $\mathfrak{p} = \mathbb{Z}[J,K]$ of $\mathbb{Z}[P,K]$.

Proposition 2.2.4. *Let $P \to Q$ be an inclusion of finitely generated torsion-free monoids, let K be an ideal of Q, let $K' = K \cap P$, let N be a combinatorial $\mathbb{Z}[Q,K]$-module, and let E be a $\mathbb{Z}[P,K']$-module. If E has t-flat dimension d relative to K, then*

$$\mathrm{Tor}_i^{\mathbb{Z}[P,K']}(N,E) = 0 \textit{ for all } i > d$$

Proof. It suffices to prove that if E is t-flat relative to K, then $\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(N,E) = 0$ (For $d > 0$, apply the t-flat case to the dth syzygy module of E). According to Proposition 1.5.4, N is a direct sum of combinatorial $\mathbb{Z}[P,K']$-modules. Since tor functors commute with direct sums, $\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(N,E)$ is the direct sum of modules of the form $\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(N',E)$ where each N' is a combinatorial $\mathbb{Z}[P,K']$-module. Each $\mathrm{Tor}_1^{\mathbb{Z}[P,K']}(N',E) = 0$ by the equivalence of (1) and (2) in Theorem 2.2.2. $\square$

Proposition 2.2.5. *Let $P \to Q$ be an inclusion of finitely generated torsion-free monoids, let E be a $\mathbb{Z}[P,K]$-module, and let $K' = K + Q$. If E has t-flat dimension d relative to K, then the $\mathbb{Z}[Q,K']$-module $E' = \mathbb{Z}[Q,K'] \otimes_{\mathbb{Z}[P,K]} E$ has t-flat dimension less than or equal to d relative to K'.*

Proof. It suffices to prove that if E is t-flat relative to K, then E' is t-flat relative to K' (For $d > 0$, apply the t-flat case to the dth syzygy module of E). Now suppose E is t-flat.

Let N be a combinatorial $\mathbb{Z}[Q,K']$-module and let $0 \to L \to F \to N \to 0$ be an exact sequence of $\mathbb{Z}[Q,K']$-modules with F free. Tensor this exact sequence with E' to get the long exact sequence

$$\cdots \to 0 \to \mathrm{Tor}_1^{\mathbb{Z}[Q,K']}(N,E') \to L \otimes_{\mathbb{Z}[Q,K']} E' \to F \otimes_{\mathbb{Z}[Q,K']} E' \to N \otimes_{\mathbb{Z}[Q,K']} E' \to 0.$$

Our long exact sequence can also be written as

$$\cdots \to 0 \to \mathrm{Tor}_1^{\mathbb{Z}[Q,K']}(N,E') \to L \otimes_{\mathbb{Z}[P,K]} E \to F \otimes_{\mathbb{Z}[P,K]} E \to N \otimes_{\mathbb{Z}[P,K]} E \to 0$$

since, for any $\mathbb{Z}[Q,K']$-module E'',

$$E'' \otimes_{\mathbb{Z}[Q,K']} E' = E'' \otimes_{\mathbb{Z}[Q,K']} \mathbb{Z}[Q,K'] \otimes_{\mathbb{Z}[P,K]} E \cong E'' \otimes_{\mathbb{Z}[P,K]} E$$

On the other hand, we have the long exact sequence obtained by taking the tensor product of our exact sequence with E over $\mathbb{Z}[P,K]$:

$$\cdots \to \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(F,E) \to \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(N,E) \to L \otimes_{\mathbb{Z}[P,K]} E \to F \otimes_{\mathbb{Z}[P,K]} E \to N \otimes_{\mathbb{Z}[P,K]} E \to 0.$$

By Proposition 2.2.4 and the fact that tor functors commute with direct sums,

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(F,E)=0.$$

Therefore,

$$\mathrm{Tor}_1^{\mathbb{Z}[Q,K']}(N,E')\cong \mathrm{Tor}_1^{\mathbb{Z}[P,K]}(N,E)=0$$

and E' is t-flat relative to K'. $\square$

Definition 2.2.6. We say a $\mathbb{Z}[P,K]$-module E is *weakly t-flat relative to* K if

$$\mathrm{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,P^+],E)=0.$$

Lemma 2.2.7. *Continuing the notation from Theorem 2.2.2, let R be a Noetherian B-algebra, let E be a finitely generated R-module and let $\widehat{E}$ be the I-adic completion of E. Then E is weakly t-flat relative to K if and only if $\widehat{E}$ is t-flat relative to K.*

Proof. First, suppose E is weakly t-flat relative to K. Since R is Noetherian, so is its I-adic completion $\widehat{R}$. Furthermore, $I\widehat{R}\subseteq \mathrm{rad}(\widehat{R})$. Consider the long exact tor sequence

$$\cdots\to 0\to \mathrm{Tor}_1^B(B_0,E)\to I\otimes_B E\to E\to B_0\otimes_B E\to 0.$$

It is an exact sequence of R-modules and $\widehat{R}$ is a flat R-module. So, after taking the tensor product with $\widehat{R}$, the resultant sequence is exact. In particular,

$$\mathrm{Tor}_1^B(B_0,\widehat{E})\cong \mathrm{Tor}_1^B(B_0,E)\otimes_R \widehat{R}=0.$$

We obtain the t-flatness of $\widehat{E}$ by the equivalence of (1) and (3') in Theorem 2.2.2.

On the other hand, suppose $\mathrm{Tor}_1^B(B_0,\widehat{E})=0$. Let S be the multiplicatively closed set $1+IR$, then $\widehat{R}$ is faithfully flat over $S^{-1}R$ and

$$\mathrm{Tor}_1^B(B_0,E)\otimes_R \widehat{R}\cong \mathrm{Tor}_1^B(B_0,\widehat{E})=0.$$

So, $\mathrm{Tor}_1^B(B_0,E)\otimes_R S^{-1}R=0$. But,

$$\mathrm{Supp}(\mathrm{Tor}_1^B(B_0,E))\subseteq \{\mathfrak{p}\subset \mathbb{Z}[P,K]\mid I\subseteq \mathfrak{p}\}.$$

That is, $\mathrm{Tor}_1^B(B_0,E)=0$. $\square$

Lemma 2.2.8. *Let $K\subseteq J$ be proper ideals of P and let E be a $\mathbb{Z}[P,K]$-module. If E is weakly t-flat relative to K, then E/JE is weakly t-flat relative to J.*

Proof. Continue the notation of Theorem 2.2.2. By the equivalence of (3) and (3') in Theorem 2.2.2, it suffices to prove that the canonical surjective homomorphism

$$(I/I(J)) \otimes_{B/JB} (E/JE) \to IE/JE$$

is an isomorphism. Since

$$(I/I(J)) \otimes_B E \cong (I/I(J)) \otimes_{B/JB} (B/JB) \otimes_B E \cong (I/I(J)) \otimes_{B/JB} (E/JE),$$

it suffices to prove that the canonical map $\varphi : (I/I(J)) \otimes_B E \to E/JE$ is injective. Take the tensor product of the short exact sequence

$$0 \to I/I(J) \to B/JB \to B_0 \to 0$$

with E over B to see $\ker\varphi \cong \mathrm{Tor}_1^B(B_0, E)$. Since E is weakly t-flat relative to K, $\ker\varphi = 0$. □

Lemma 2.2.9. *Let K, K_1 and K_2 be proper ideals of P, let $K_1 \cap K_2 \subseteq K$, and let E be a $\mathbb{Z}[P, K]$-module. If E/K_1E is weakly t-flat relative to K_1 and E/K_2E is weakly t-flat relative to K_2, then E is weakly t-flat relative to K.*

Proof. We continue the notation from Theorem 2.2.2. It suffices to prove the multiplication map

$$(I^n/I^{n+1}) \otimes_{B_0} E_0 \to I^nE/I^{n+1}E$$

is an isomorphism for all n by the equivalence of (3) and (7). For any particular n, we may assume $(n+1)P^+ \subseteq K$ when attempting to prove this map is an isomorphism. From now on, assume $(n+1)P^+ \subseteq K$. By the equivalence of (3) and (3') it suffices to prove $I \otimes E \to E$ is injective. Take the tensor product of E with the short exact sequence

$$0 \to I \to I/I(K_1) \oplus I/I(K_2) \to I/I(K_1 \cup K_2) \to 0 \tag{2.1}$$

and consider the commutative diagram

$$\begin{array}{ccccccc} I \otimes E & \xrightarrow{g} & (I/I(K_1)) \otimes E \oplus (I/I(K_2)) \otimes E & \to & (I/I(K_1 \cup K_2)) \otimes E & \to & 0 \\ f\downarrow & & h\downarrow & & & & \\ E & \xrightarrow{j} & E/K_1E \oplus E/K_2E & & & & \end{array} \tag{2.2}$$

We want to prove the f is injective. Since E/K_1E is weakly t-flat relative to K_1 and E/K_2E is weakly t-flat relative to K_2, h is injective. Since h is injective, it suffices to prove g is

injective, then $h \circ g = j \circ f$ is injective and f is injective. We will prove g is injective by induction on n. If $n = 1$, $K = K_1 = K_2 = P^+$ and we are done. Now suppose E/JE is weakly t-flat whenever $nP^+ \cup (K_1 \cap K_2) \subseteq J$. In particular, let $J = K \cup nP^+$. Notice that I, $I/I(K_1) \oplus I/I(K_2)$ and $I/I(K_1 \cup K_2)$ are $\mathbb{Z}[P, J]$-modules since I is annihilated by $I(J)$. Furthermore, the short exact sequence (2.1) above is an exact sequence of $\mathbb{Z}[P, J]$-modules. In fact, the top line of our commutative diagram (2.2) can be obtained by taking the tensor product over $\mathbb{Z}[P, J]$ of our short exact sequence (2.1) above with E/JE. So,

$$\mathrm{Tor}_1^B(I/I(K_1 \cup K_2), E) = \mathrm{Tor}_1^{\mathbb{Z}[P,J]}(I/I(K_1 \cup K_2), E/JE)$$

Since E/JE is weakly t-flat relative to J and $I/I(K_1 \cup K_2)$ is a combinatorial $\mathbb{Z}[P, J]$-module, we have $\mathrm{Tor}_1^{\mathbb{Z}[P,J]}(I/I(K_1 \cup K_2), E/JE) = 0$. Therefore, g is an injection. □

Proposition 2.2.10. *Let R be a Noetherian $\mathbb{Z}[P]$-algebra and let E be a finitely generated R-module. There exists an ideal $K \subseteq P^+$ such that the module E/JE is weakly t-flat relative to an ideal J if and only if $K \subseteq J$.*

Proof. Let $S = \{J \mid E/JE$ is weakly t-flat relative to $J\}$, let $K = \bigcap_{J \in S} J$, and let $I = \mathbb{Z}[P^+, K]$. By Lemma 2.2.8, it suffices to prove E/KE is weakly t-flat relative to K, for then $S = \{J \mid K \subseteq J\}$. By the equivalence of (3') and (7) in Theorem 2.2.2, to prove E/KE is weakly t-flat relative to K, it suffices to prove $E/I^{n+1}E$ is t-flat relative to $(n + 1)P^+ \cup K$ for all $n \in \mathbb{N}$. If $J \in S$, then $E/(I^{n+1} + I(J))E$ is t-flat relative to $(n + 1)P^+ \cup J$ for all $n \in \mathbb{N}$ by the same equivalence. That is, $(n + 1)P^+ \cup J \in S$ for all $n \in \mathbb{N}$. Furthermore, $(n + 1)P^+ \cup K = \bigcap_{J \in S}((n + 1)P^+ \cup J)$. Since there are only finitely many ideals of P containing $(n + 1)P^+$, this intersection is finite and $E/I^{n+1}E$ is t-flat relative to $(n + 1)P^+ \cup K$ by Lemma 2.2.9. □

2.3 Openness of t-Flat Loci

Definition 2.3.1. (Ogus [33, Definition 2.4]) Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a locally Noetherian coherent log scheme. We say a sheaf of ideals $\mathcal{K} \subseteq \mathcal{M}_X$ is *coherent* if, locally on X, there exists a fine chart $P_X \to \mathcal{M}_X$ and an ideal $K \subseteq P$ such that $\mathcal{K}$ is the ideal sheaf generated by the image of K.

Proposition 2.3.2. *(Ogus [33, Proposition 2.6]) Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a locally Noetherian fine log scheme, let $\mathcal{K} \subseteq \mathcal{M}_X$ be a coherent sheaf of ideals, let $\beta : P_X \to \mathcal{M}_X$ be*

a fine chart, and let $K = \beta^{-1}(\mathcal{K})$. Then $\mathcal{K}$ is the ideal sheaf generated by the image of K.

Theorem 2.3.3. *Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a locally Noetherian toroidal log scheme, let $\mathcal{K} \subseteq \mathcal{M}_X$ be a coherent ideal sheaf, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}(\mathcal{K})$. Then*

$$\{x \in X \mid \mathcal{F} \textit{ is } \mathcal{M}_X\textit{-flat relative to } \mathcal{K} \textit{ at } x\}$$

is open.

Proof. The question is local. Fix $x \in X$. By Proposition 1.3.3, we may assume $X = \operatorname{Spec}(P \xrightarrow{\beta} R)$ where $P = \overline{\mathcal{M}}_{X,x}$, R is a Noetherian ring and $x = \mathfrak{p}$. In particular, β is local at x and we have an associated ring homomorphism $\widetilde{\beta} : \mathbb{Z}[P] \to R$. By the previous proposition, we may also assume $\mathcal{K}$ is generated by the image of an ideal $K \subsetneq P$. Furthermore, we may assume $\mathcal{F}$ is the sheaf associated to some finitely generated R-module E.

By Corollary 2.1.4, $\mathcal{F}$ is $\mathcal{M}_X$-flat relative to $\mathcal{K}$ at $x \in X$ if and only if $\mathcal{F}_x$ is t-flat relative to K. So, $\mathcal{F}$ is not $\mathcal{M}_X$-flat relative to $\mathcal{K}$ at $x \in X$ if and only if $\widetilde{\beta}^{-1}(\mathfrak{p})$ is in the support of $\operatorname{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J], E)$ for some ideal $J \subset P$ containing K. Since $\mathbb{Z}[P,J]$ is a finitely generated combinatorial $\mathbb{Z}[P]$-module, in order to check

$$\operatorname{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J], E) = 0 \quad \forall J \subset P \text{ containing } K,$$

it suffices to check only the prime ideals $J \subset P$ containing K by Lemma 2.2.3. Since P is finitely generated, P has only finitely many primes. Therefore,

$$\{\operatorname{Tor}_1^{\mathbb{Z}[P,K]}(\mathbb{Z}[P,J], E) \mid K \subseteq J \subset P,\ J \text{ prime}\}$$

is a finite set of finitely generated modules. So, the union of the supports of these modules is closed. But x is in this union if and only if $\mathcal{F}$ is not $\mathcal{M}_X$-flat relative to $\mathcal{K}$ at x. So,

$$\{x \in X \mid \mathcal{F} \text{ is } \mathcal{M}_X\text{-flat relative to } \mathcal{K} \text{ at } x\}$$

is open. $\square$

Proposition 2.3.4. *Let P be a finitely generated torsion-free monoid, let R be a Noetherian ring, let $\beta : P \to R$ be a monoid homomorphism with respect to multiplication, let E be a finitely generated R-module, and let $K_E : \operatorname{Spec} R \to \{\textit{ideals of } P\}$ be the function that takes a prime $\mathfrak{p}$ to the ideal $K_E(\mathfrak{p})$ of P such that $E_{\mathfrak{p}}$ is weakly t-flat relative to J if and only if $K_E(\mathfrak{p}) \subseteq J$. Then, the image of K_E is finite.*

Proof. We will prove this by Noetherian induction. It suffices to prove that K_E is constant on some nonempty open subset of $\operatorname{Spec} R$. Let $\{X_i\}_{i=1}^n$ be the set of irreducible components of $\operatorname{Spec} R$, let $X = X_1 \setminus \bigcup_{i=2}^n X_i$ and let η be the generic point of X. Let K be the ideal of P such that E_η is weakly t-flat relative to J if and only if $K \subseteq J$. Now consider $X_K = \{x \in X \mid K_E(x) = K\}$. Since η is in X_K, X_K is nonempty. For each point x on X, let $U(x) = \{x' \in X \mid E_{x'}$ is weakly t-flat relative to $K_E(x)\}$. Note that $U(x) = \{x' \in X \mid K_E(x') \subseteq K_E(x)\}$. By Theorem 2.3.3, $U(x)$ is open for all x. Since each $U(x)$ is open, η is an element of $U(x)$ for all x in X. That is, $K_E(\eta) \subseteq K_E(x)$ for all x in X. So, $X_K = U(\eta)$ and X_K is open. $\square$

Example 2.3.5. Let $P = \mathbb{N}^2$, let $R = \mathbb{C}[x, y]$, let $\beta : P \to R$ be given by $(n, m) \mapsto x^n y^m$, and let $E = R/(x - y) \bigoplus R/(y^3)$. Here

$$K_E(\mathfrak{p}) = \begin{cases} P^+, & \text{if } \mathfrak{p} = (x, y); \\ (0,3) + P & \text{if } \mathfrak{p} = (x - \alpha, y) \text{ with } \alpha \neq 0 \text{ or } \mathfrak{p} = (y); \\ \varnothing & \text{otherwise.} \end{cases}$$

Chapter 3

Log Blowing Up and Log Flattening

3.1 Definition of a Log Blowup

If P is a finitely generated monoid and K is an ideal of P, then the log blowup of $X = \operatorname{Spec} \mathbb{Z}[P]$ along the coherent ideal $\mathcal{K}$ generated by the image of K in $\mathcal{M}_X$ is the log scheme whose underlying scheme is the blowup of X along the coherent sheaf of ideals generated by the image of K in $\mathcal{O}_X$:

$$\operatorname{Proj} \bigoplus_{n \geq 0} \mathbb{Z}[nK]$$

and whose log structure is given locally by the canonical log structure $P + \mathbb{N}(K - p) \to \mathbb{Z}[P + \mathbb{N}(K - p)]$ on the affine open subscheme $\operatorname{Spec} \mathbb{Z}[P + \mathbb{N}(K - p)] \subseteq \operatorname{Proj} \bigoplus_{n \geq 0} \mathbb{Z}[nK]$ for each $p \in K$. Call this log scheme $\operatorname{Proj}(\bigsqcup_{n \geq 0} nK \to \bigoplus_{n \geq 0} \mathbb{Z}[nK])$.

More generally, let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a fine log scheme. If P is a finitely generated monoid, if K is an ideal of P, and if $U = \operatorname{Spec}(P \xrightarrow{\beta} R)$ is an open affine subscheme of X, then the log blowup $\pi : \operatorname{Bl}_{\mathcal{K}} X \to X$ of X along the coherent sheaf of ideals $\mathcal{K} \subseteq \mathcal{M}_X$ generated by the image of K has

$$\pi^{-1}(U) = U \times_{\operatorname{Spec}(P \to \mathbb{Z}[P])} \operatorname{Proj}(\bigsqcup_{n \geq 0} nK \to \bigoplus_{n \geq 0} \mathbb{Z}[nK])$$

equipped with the log structure given by pullback from $\operatorname{Proj}(\bigsqcup_{n \geq 0} nK \to \bigoplus_{n \geq 0} \mathbb{Z}[nK])$.

Proposition 3.1.1. *(Universal Property of Log Blowing Up) Let $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ be a fine log scheme and let $\mathcal{K}$ be a coherent ideal of $\mathcal{M}_X$. If $f : (Z, \alpha_Z : \mathcal{M}_Z \to \mathcal{O}_Z) \to$*

$(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is any morphism of fine log schemes such that $f^*\mathcal{K}$ is invertible, then f factors uniquely through $\pi : \mathrm{Bl}_{\mathcal{K}} X \to X$ the log blowup of X along $\mathcal{K}$.

Proof. The question is local on X so we may assume $X = \mathrm{Spec}(P \xrightarrow{\beta} R)$ for some finitely generated monoid P and $\mathcal{K}$ is generated by the image of some ideal $K \subseteq P$. Since the log blowup is defined by pullback from $\mathrm{Spec}(P \to \mathbb{Z}[P])$ and the pullback of an invertible ideal is invertible, we may assume $X = \mathrm{Spec}(P \to \mathbb{Z}[P])$. Moreover, by Proposition 2.3.2, we may assume $\mathcal{K}$ is generated by the image of some ideal $K \subseteq P$. Let $\mathrm{Spec}(Q \xrightarrow{\gamma} R')$ be an affine open subscheme of Z with Q finitely generated. We have a commutative diagram

$$\begin{array}{ccc} P & \hookrightarrow & \mathbb{Z}[P] \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ Q & \underset{\gamma}{\to} & R' \end{array}$$

with $\varphi(K) + Q$ principal. So, there exists some $p \in K$ such that $\varphi(p) + Q = \varphi(K) + Q$. That is, $\varphi^{gp}(K - p) \subseteq Q$. Therefore, φ factors through $P + \mathbb{N}(K - p)$. Since the monoid homomorphism $P \hookrightarrow P + \mathbb{N}(K - p)$ is an epimorphism, this factorization is unique. $\square$

Lemma 3.1.2. *[19, Lemma 3.10] Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a fine log scheme, let $\mathcal{K}$ and $\mathcal{K}'$ be coherent sheaves of ideals in $\mathcal{M}_X$, and let $\pi : \mathrm{Bl}_{\mathcal{K}} X \to X$ be the log blowup of $\mathcal{K}$. Then, $\mathrm{Bl}_{\pi^{-1}\mathcal{K}'}(\mathrm{Bl}_{\mathcal{K}} X) \cong \mathrm{Bl}_{\mathcal{K}+\mathcal{K}'} X$.*

Proof. The question is local on X so we may assume $X = \mathrm{Spec}(P \xrightarrow{\beta} R)$ for some finitely generated monoid P and some ring R. Since the log blowup is defined by pullback from $\mathrm{Spec}(P \to \mathbb{Z}[P])$, we may assume $X = \mathrm{Spec}(P \to \mathbb{Z}[P])$. Moreover, by Proposition 2.3.2, we may assume $\mathcal{K}$ is generated by the image of some ideal $K \subseteq P$ and $\mathcal{K}'$ is generated by the image of $K' \subseteq P$. Now, $\mathrm{Bl}_{\mathcal{K}+\mathcal{K}'} X$ is covered by open affines of the form

$$U = \mathrm{Spec}(P + \mathbb{N}((K + K') - (p + p')) \to \mathbb{Z}[P + \mathbb{N}((K + K') - (p + p'))])$$

for some $p \in K$ and $p' \in K'$. Since

$$P + \mathbb{N}((K + K') - (p + p')) = (P + \mathbb{N}(K - p)) + \mathbb{N}((P + \mathbb{N}(K - p) + K') - p'),$$

U is also an open affine subscheme of $\mathrm{Bl}_{\pi^{-1}\mathcal{K}'}(\mathrm{Bl}_{\mathcal{K}} X)$. It is the open affine given by $p' \in P + \mathbb{N}(K - p) + K'$ which lies over the open affine of $\mathrm{Bl}_{\mathcal{K}} X$ given by $p \in K$. Furthermore,

in $\mathrm{Bl}_{\mathcal{K}+\mathcal{K}'} X$ we have

$$\begin{aligned}\operatorname{Spec}\mathbb{Z}[P+\mathbb{N}((K+K')-(p+p'))] &\cap \operatorname{Spec}\mathbb{Z}[P+\mathbb{N}((K+K')-(q+q'))] \\ &= \operatorname{Spec}\mathbb{Z}[P+\mathbb{N}((K+K')-(p+p'))+\mathbb{N}((K+K')-(q+q'))].\end{aligned}$$

Since

$$\begin{aligned}P&+\mathbb{N}((K+K')-(p+p'))+\mathbb{N}((K+K')-(q+q')) \\ &= (P+\mathbb{N}(K-p)+\mathbb{N}(K-q))+\mathbb{N}((P+\mathbb{N}(K-p)+\mathbb{N}(K-q)+K')-p') \\ &\qquad + \mathbb{N}((P+\mathbb{N}(K-p)+\mathbb{N}(K-q)+K')-q'),\end{aligned}$$

these pieces are glued together in the same way in both $\mathrm{Bl}_{\mathcal{K}+\mathcal{K}'} X$ and $\mathrm{Bl}_{\pi^{-1}\mathcal{K}'}(\mathrm{Bl}_{\mathcal{K}} X)$. □

3.2 Generalized Toric Varieties

Definition 3.2.1. Let G be a finitely generated free Abelian group. We say Δ is a *cofan* in G if Δ is a finite set of finitely generated submonoids of G such that:

(1) For each $P \in \Delta$, $P^{gp} = G$.

(2) If $P \in \Delta$, every localization of P is also an element of Δ.

(3) If $P, Q \in \Delta$, then $P+Q$ is a localization of both P and Q.

Given a cofan Δ, we may form a (not necessarily normal) toric variety $X(\Delta)$ (over $\mathbb{Z}$) as follows:

(1) For each each monoid $P \in \Delta$, we form the affine log scheme associated to the canonical monoid homomorphism $P \to \mathbb{Z}[P]$. That is, form $\operatorname{Spec}(P \to \mathbb{Z}[P])$.

(2) For each pair of monoids $P, Q \in \Delta$, glue $\operatorname{Spec}(P \to \mathbb{Z}[P])$ to $\operatorname{Spec}(Q \to \mathbb{Z}[Q])$ along their common open subscheme $\operatorname{Spec}(P+Q \to \mathbb{Z}[P+Q])$.

In particular, if P is a finitely generated torsion-free monoid, every log blowup of $\operatorname{Spec}(P \to \mathbb{Z}[P])$ is a toric variety in this sense. Given a ring R, we will also consider the toric variety over R, $X(\Delta)_R = \operatorname{Spec} R \times X(\Delta)$.

Remark 3.2.2. If Σ is a fan (see Fulton [9]), then $\Delta = \{S_\sigma \mid \sigma \in \Sigma\}$ is a cofan. Furthermore, if Δ is a cofan, then the set of dual cones of the cones in $G \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the elements of Δ is a fan in $\operatorname{Hom}(G, \mathbb{Z})$.

Given any cofan Δ, we (in the spirit of DeMeyer, Ford & Miranda [6]) make Δ into a topological space by letting the open sets be the subsets of Δ that are also cofans. Given a finitely generated torsion-free monoid P, the set of localizations of P is a cofan. In fact, the topological space associated to the localizations of P is homeomorphic to the set of prime ideals of P equipped with the topology whose closed sets are of the form $\{\mathfrak{p} \mid J \subseteq \mathfrak{p}\}$ for some ideal $J \subseteq P$. Of course, such a topological space supports a canonical sheaf of monoids, where the sections of this sheaf on the open subset given by some localization Q of P is Q itself. In light of these considerations, our construction of a toric variety from a cofan is a special case of a more general construction that takes a topological space with a distinguished sheaf of monoids that is locally isomorphic to spectra of monoids to a log scheme over some affine base scheme $\operatorname{Spec} R$ by extending the functor that takes the set of localizations of P with its canonical sheaf of monoids to $\operatorname{Spec}(P \to R[P])$. We write $\mathcal{M}_\Delta$ for the distinguished sheaf of monoids (structure sheaf) on Δ.

A morphism of cofans $\Delta' \to \Delta$ is a pair $(\varphi, \varphi^\#)$ consisting of a continuous map $\varphi : \Delta' \to \Delta$ of the underlying topological spaces and a local homomorphism of sheaves of monoids $\varphi^\# : \varphi^{-1}(\mathcal{M}_\Delta) \to \mathcal{M}_{\Delta'}$. In particular, if Δ is the set of localizations of some finitely generated torsion-free monoid P and $K \subseteq P$ is an ideal, then the blowup $\operatorname{Bl}_K X(\Delta) \cong X(\Delta')$ where Δ' is the set $\{Q_p \mid p \in K\}$ and Q_p is the submonoid of P^{gp} generated by P and $K - p$. The inclusions $P \hookrightarrow Q_p$ induce a morphism of cofans $\Delta' \to \Delta$ and the map $\pi : \operatorname{Bl}_K X(\Delta) \to X(\Delta)$ is the induced map $X(\Delta') \to X(\Delta)$.

Let Δ be a cofan and let $P \in \Delta$. To P we associate the open affine subscheme $U_P = \operatorname{Spec} \mathbb{Z}[P] \subseteq X(\Delta)$ and its (locally) closed subscheme $Z_P = \operatorname{Spec} \mathbb{Z}[P, P^+] \cong \operatorname{Spec} \mathbb{Z}[P^*]$. The set $\{Z_P \mid P \in \Delta\}$ is a stratification of $X(\Delta)$. Similarly, if R is a ring, we write $(U_P)_R = \operatorname{Spec} R[P] \subseteq X(\Delta)_R$ and $(Z_P)_R = \operatorname{Spec} R[P, P^+] \cong \operatorname{Spec} R[P^*]$. The set $\{(Z_P)_R \mid P \in \Delta\}$ is a stratification of $X(\Delta)_R$.

3.3 The Flattening Theorem

Proposition 3.3.1. *Let P be a finitely generated torsion-free monoid, let R be a Noetherian ring, let $X = \operatorname{Spec}(P \hookrightarrow R[P])$, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, let $\mathcal{K} \subseteq \mathcal{M}_X$ be a coherent sheaf of ideals and let $\pi : \operatorname{Bl}_{\mathcal{K}} X \to X$ be the log blowup. If $\mathcal{F}$ has t-flat dimension $\leq d$, then $\pi^*(\mathcal{F})$ has t-flat dimension $\leq d$.*

Proof. This follows immediately from Proposition 2.2.5 and Proposition 2.3.2. □

Definition 3.3.2. Let D be the exceptional divisor of $\pi : \mathrm{Bl}_{\mathcal{K}} X \to X$. We say $\mathcal{F}' = \pi^*(\mathcal{F})/\underline{\Gamma}_D(\pi^*(\mathcal{F}))$ is the strict transform of $\mathcal{F}$.

Lemma 3.3.3. *Let P be a finitely generated torsion-free monoid, let R be a Noetherian ring, let $X = \mathrm{Spec}(P \hookrightarrow R[P])$, let x be a point of the closed subscheme $R[P, P^+]$, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules that is not t-flat at x, let K be the ideal of P that captures to failure of $\mathcal{F}_x$ to be t-flat (see Proposition 2.2.10), and let $\mathcal{K} \subseteq \mathcal{M}_X$ be the ideal generated by the image of K. If D is the exceptional divisor of $\pi : \mathrm{Bl}_{\mathcal{K}} X \to X$, $y \in \mathrm{Bl}_{\mathcal{K}} X$ lies over x, Z is the stratum of $\mathrm{Bl}_{\mathcal{K}} X$ containing y, and $\mathcal{F}'$ is the strict transform of $\mathcal{F}$, then the map*

$$\varphi : \pi^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathrm{Bl}_{\mathcal{K}} X}} \mathcal{O}_{Z,y} \to \mathcal{F}' \otimes_{\mathcal{O}_{\mathrm{Bl}_{\mathcal{K}} X}} \mathcal{O}_{Z,y}$$

is not an isomorphism.

Proof. Let $\{p_i\}_{i=1}^n$ be a minimal generating set for K. Without loss of generality, we may assume y is in the open subset $\mathrm{Spec}\, R[P + K - p_1]$ of $\mathrm{Bl}_{\mathcal{K}} X$. Let $J = (p_1 + P^+) \cup (\bigcup_{i=2}^n (p_i + P)) = K \setminus p_1 + P^*$, let $I = \mathbb{Z}[P^+]$ and let $E = \widehat{\mathcal{F}_x}$. (During the rest of this proof, we will write K for $\mathbb{Z}[K]$ and J for $\mathbb{Z}[J]$, etc.)

As a first step, we will prove there exist $e_1, e_2, \ldots, e_n \in \mathcal{F}_x$ such that $e_1 \in \mathcal{F}_x \setminus I\mathcal{F}_x$ and $\sum_{i=1}^n t^{p_i} e_i = 0$. To do this, it suffices to prove there exists $e_1 \in \mathcal{F}_x$ such that $e_1 \in \mathcal{F}_x \setminus I\mathcal{F}_x$ and $t^{p_1} e_1 \in J\mathcal{F}_x$. For then, we can find $e_1', e_2', \ldots, e_n'$ with $e_1' \in I\mathcal{F}_x$ such that $\sum_{i=1}^n t^{p_i} e_i' = t^{p_1} e_1$ and the elements $e_1' - e_1, e_2', \ldots, e_n'$ are our desired elements. In fact, it suffices to find $e_1 \in \mathcal{F}_x$ such that $e_1 \in E \setminus IE$ and $t^{p_1} e_1 \in JE$. For then, we consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J\mathcal{F}_x & \to & \mathcal{F}_x & \to & \mathcal{F}_x/J\mathcal{F}_x & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J\mathcal{F}_x \otimes_{\mathbb{Z}[P]} \mathbb{Z}[[P]] & \to & E & \to & E/JE & \to & 0 \end{array}$$

where the nontrivial vertical maps are injections and $J\mathcal{F}_x \otimes_{\mathbb{Z}[P]} \mathbb{Z}[[P]] = JE$ by faithful flatness since x is a point on the closed subscheme $R[P, P^+]$. We know that any element of $t^{p_1} e_1 \in JE \cap \mathcal{F}_x$ goes to zero in E/JE. So, it must also go to zero in $\mathcal{F}_x/J\mathcal{F}_x$. That is, $t^{p_1} e_1 \in J\mathcal{F}_x$.

By Lemma 2.2.7 applied to an ideal $J' \subseteq P$ containing J, $E/J'E$ is (weakly) t-flat relative to J' if and only if $p_1 \in J'$. Consider the commutative square

$$\begin{array}{ccc} \mathbb{Z}[P,J] \otimes_{\mathbb{Z}[P,P^+]} E/IE & \xrightarrow[\text{surjection}]{\text{canonical}} & E/JE \\ \downarrow & & \downarrow \\ \mathbb{Z}[P,K] \otimes_{\mathbb{Z}[P,P^+]} E/IE & \xrightarrow{\text{isomorphism}} & E/KE. \end{array}$$

We know the upper map is not an isomorphism. If it were, E would be t-flat relative to J by (4')⇒(1) in Theorem 2.2.2. An element of the kernel of the upper map must also be an element of the kernel of the left-hand map. The kernel of the left-hand map is $t^{p_1} \otimes (E/IE)$. So, every nonzero element of the kernel of the upper map is of the form $t^{p_1} \otimes e$. Let $e_1 \in \mathcal{F}_x$ be a lift of e for some particular nonzero element $t^{p_1} \otimes e$ of the kernel of the upper map. Since $t^{p_1} \otimes e$ is in the kernel of the upper map, $t^{p_1} e_1 \in JE$.

So, there exist elements $e_1, e_2, \ldots, e_n$ of $\mathcal{F}_x$ such that $t^{p_1}e_1 + t^{p_2}e_2 + \cdots + t^{p_n}e_n = 0$ and $e_1 \notin I\mathcal{F}_x$. Notice that the image of $e_1 + \sum_{i=2}^n t^{p_i - p_1} e_i \in \pi^*(\mathcal{F})_y$ in $\mathcal{F}'_y$ is zero since it is annihilated by the image of t^{p_1}. Considering $\sum_{i=1}^n e_i \otimes t^{p_i - p_1} \in \pi^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathrm{Bl}_{\mathcal{K}} X}} \mathcal{O}_{Z,y}$, we see that this element is in the kernel of φ. So, it suffices to prove $\sum_{i=1}^n e_i \otimes t^{p_i - p_1}$ is not zero in $\pi^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathrm{Bl}_{\mathcal{K}} X}} \mathcal{O}_{Z,y}$.

We will prove $\sum_{i=1}^n e_i \otimes t^{p_i - p_1}$ is not zero in $\mathcal{F}_x \otimes_{R[P]} \mathcal{O}_{Z,y} = \pi^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathrm{Bl}_{\mathcal{K}} X}} \mathcal{O}_{Z,y}$ by contradiction.. According to Eisenbud [7, Lemma 6.4], if $\sum_{i=1}^n e_i \otimes t^{p_i - p_1}$ were zero in $\mathcal{F}_x \otimes_{R[P]} \mathcal{O}_{Z,y}$, then there would exist $e'_j \in \mathcal{F}_x$ and $a_{ij} \in R[P]$ such that $\sum_j a_{ij} e'_j = e_i$ for all i and $\sum_{i=1}^n a_{ij} t^{p_i - p_1} = 0$ in $\mathcal{O}_{Z,y}$. Since $e_1 \notin R[P^+]\mathcal{F}_x$, there would exist a j such that $a_{1j} \notin R[P^+]$. We may assume this j is 1. Now we would have $a_{11} \notin R[P^+]$ and $a_{11} + \sum_{i=2}^n a_{i1} t^{p_i - p_1} = 0$ in $\mathcal{O}_{Z,y}$. So, there would exist a $p \in P^*$ such that the degree p homogeneous piece of a_{11} is not zero. Looking at degree p pieces, we would see that at least one of the a_{i1} has a nonzero degree $p + p_1 - p_i$ degree piece. So, we would conclude $p + p_1 - p_i \in P$. Since p is a unit in P, we would have $p_1 - p_i \in P$. But, this contradicts the minimality of $\{p_1, p_2, \ldots, p_n\}$. So, $\sum_{i=1}^n e_i \otimes t^{p_i - p_1}$is not zero in $\pi^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathrm{Bl}_{\mathcal{K}} X}} \mathcal{O}_{Z,y}$. □

Theorem 3.3.4. *(Flattening Theorem) Let R be a Noetherian ring, let $X = X(\Delta)_R$ be a toric variety over R, and let $\mathcal{F}$ be a coherent sheaf on $X(\Delta)_R$.*

(1) There is a log blowup $\widetilde{X}$ of X with projection $\pi : \widetilde{X} \to X$ such that $\pi^*(\mathcal{F})$ has t-tor dimension ≤ 1.

(2) There is a log blowup $\widetilde{X}$ of X with projection $\pi : \widetilde{X} \to X$ such that the strict transform of $\mathcal{F}$ is t-flat.

Proof. First we will prove (2) implies (1). Let D be the exceptional divisor and write an exact sequence $0 \to \mathcal{N}_{\mathcal{F}} \to \mathcal{O}_X^n \to \mathcal{F} \to 0$. On $\widetilde{X}$ we have

$$0 \to \pi^*(\mathcal{N}_{\mathcal{F}})/\underline{\Gamma}_D(\pi^*(\mathcal{N}_{\mathcal{F}})) \to \mathcal{O}_{\widetilde{X}}^n \to \pi^*(\mathcal{F}) \to 0.$$

In particular, if the strict transform of $\mathcal{N}_{\mathcal{F}}$ is t-flat, $\pi^*(\mathcal{F})$ has t-tor dimension ≤ 1.

We now prove (2). For each $x \in X$ such that $\mathcal{F}_x$ is t-flat, let $(Z_P)_R$ be the stratum of X containing x, let $U_P = \operatorname{Spec} \mathbb{Z}[P]$, let $(U_P)_R = \operatorname{Spec} R[P]$, let $\mathcal{K}_{\mathcal{F}}(x)$ be the ideal of $\mathcal{M}_{(U_P)_R}$ that measures the failure of $\mathcal{F}_x$ to be t-flat (see Proposition 2.2.10), and let $\mathcal{I}_{\mathcal{F}}(x)$ be the ideal of $\mathcal{O}_{(U_P)_R}$ generated by the image of $\mathcal{K}_{\mathcal{F}}(x)$. Extend $\mathcal{K}_{\mathcal{F}}(x)$ to all of X by taking the scheme theoretic closure of the subscheme of $(U_P)_R$ given by $\mathcal{I}_{\mathcal{F}}(x)$ and then taking the pre-image in $\mathcal{M}$ of the ideal sheaf of this closure. By proposition 2.3.4, only finitely many ideals on X are obtained in this manner. Let $\mathcal{K} \subset \mathcal{M}$ be the sum of these ideals, let $X(\Delta^{(1)}) = \operatorname{Bl}_{\mathcal{K}} X$ with projection $\pi_1 : X(\Delta^{(1)}) \to X$, and let $\mathcal{F}^{(1)}$ be the strict transform of $\mathcal{F}$. If $\mathcal{F}^{(n)}$ (on $X(\Delta^{(n)})$) is not t-flat, repeat this process to produce $\mathcal{F}^{(n+1)}$ (on $X(\Delta^{(n+1)})$).

If the theorem did not hold, we would get an infinite tower

$$\cdots \to X(\Delta^{(2)}) \to X(\Delta^{(1)}) \to X.$$

Let $Z_n \subset X(\Delta^{(n)})$ be the closed subscheme where $\mathcal{F}^{(n)}$ is not t-flat. By Proposition 3.3.1, Z_{n+1} maps to Z_n. Let $\Delta_{nf}^{(n)} = \{P \in \Delta^{(n)} \mid (Z_P)_R \cap Z_n \neq \emptyset\}$. $\Delta_{nf}^{(n)}$ is a finite subset of $\Delta^{(n)}$. If $\Delta_{nf}^{(n)}$ were empty, $X(\Delta^{(n)})$ would be t-flat. So, if the theorem does not hold, then $\Delta_{nf}^{(n)}$ is non-empty. Furthermore, $\Delta_{nf}^{(n+1)}$ maps to $\Delta_{nf}^{(n)}$. Hence the inverse limit of $\Delta_{nf}^{(n)}$ would not be empty if the theorem did not hold.

Fix $n \mapsto P^{(n)}$ in the inverse limit. Since $P^{(n)} \subseteq P^{(n+1)} \subseteq (P^{(n)})^{gp}$, and $\dim P^{(n)} + \operatorname{rank}(P^{(n)})^* = \operatorname{rank}(P^{(n)})^{gp}$, $n \mapsto \dim P^{(n)}$ is non-increasing. Since $n \mapsto \dim P^{(n)}$ is non-increasing, there exists an n_0 such that for $n \geq n_0$ it is constant. Here each map $(P^{(n)})^* \to (P^{(n+1)})^*$ is an injection. For $n \geq n_0$, if G is the stalk of the structure sheaf of Δ at the generic point, each unit group $(P^{(n)})^*$ has the same saturation in G (which we think of as the stalk at the generic point for all $\Delta^{(n)}$). Since $(P^{(n_0)})^*$ has finite index in its saturation in G, there exists an n_1 such that for $n \geq n_1$ the map $(P^{(n_1)})^* \to (P^{(n)})^*$ is an isomorphism.

Since each $(Z_P^{(n)})_R = \operatorname{Spec} R[P^{(n)}, (P^{(n)})^+] \cong \operatorname{Spec} R[(P^{(n)})^*]$, $(Z_P^{(n)})_R \to (Z_P^{(n_1)})_R$ is an isomorphism for $n \geq n_1$. We have coherent sheaves $\mathcal{F}^{(n)} \otimes \mathcal{O}_{(Z_P^{(n)})_R}$ on $(Z_P^{(n_1)})_R$ for $n \geq n_1$. By Lemma 3.3.3, the $(n+1)$st sheaf is a quotient of the nth sheaf by a non-zero subsheaf for all $n \geq n_1$ This contradicts the Noetherian assumption. □

Corollary 3.3.5. *Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a fine log scheme and let $\mathcal{F}$ be a coherent sheaf on X.*

(1) There is a log blowup $\widetilde{X}$ of X with projection $\pi : \widetilde{X} \to X$ such that $\pi^*(\mathcal{F})$ has t-tor dimension ≤ 1.

(2) There is a log blowup $\widetilde{X}$ of X with projection $\pi : \widetilde{X} \to X$ such that the strict transform of $\mathcal{F}$ is t-flat.

Proof. The question is local on X so we may assume $X = \operatorname{Spec}(P \xrightarrow{\beta} R)$ for some finitely generated monoid P and some ring R. By Lemma 3.3.6, we may assume P is saturated. By replacing P with $\overline{P}$, if necessary, we may also assume P is torsion-free. Summing up, we may assume X is a closed subscheme of the toric variety $\operatorname{Spec}(P \hookrightarrow R[P])$ over R. Extend $\mathcal{F}$ by zero to get a coherent sheaf on $\operatorname{Spec}(P \hookrightarrow R[P])$. Apply Theorem 3.3.4. □

Lemma 3.3.6. *Let P be a finitely generated monoid. There is a non-empty ideal $K \subseteq P$ such that $\mathrm{Bl}_K(\operatorname{Spec}\mathbb{Z}[P]) = \operatorname{Spec}\mathbb{Z}[P^{sat}]$.*

Proof. Consider the finitely generated Abelian group P^{gp}. Since P^{sat} is saturated, P^{sat} contains all the torsion elements of P^{gp}. Since the torsion subgroup of P^{gp} is a split subgroup, P is the direct sum of the torsion subgroup and the saturation of some finitely generated submonoid of the rest. The saturation of a finitely generated torsion-free monoid is finitely generated (see Rosales and García-Sánchez [34, Section 7.4]). So, P^{sat} is finitely generated since it is the sum of a finitely generated monoid and a finite group.

Since P^{sat} is finitely generated, it suffices to prove $\operatorname{Spec}\mathbb{Z}[P + \mathbb{N}p]$ is a log blowup of $\operatorname{Spec}\mathbb{Z}[P]$ for any $p \in P^{sat}$ by Lemma 3.1.2. Let $p = a - b$ with $a, b \in P$ and $np \in P$ for some positive integer n. We blowup along $(a+P) \cup (b+P)$. One open subset of the blowup is $\operatorname{Spec}\mathbb{Z}[P + \mathbb{N}p] = \operatorname{Spec}\mathbb{Z}[P][t^{a-b}]$. We want to show its complement is empty. That is, we want to show $t^{a-b} \in \mathbb{Z}[P][t^{b-a}]$. Indeed $a - b = np + (n-1)(b-a)$. □

Chapter 4

Kato's Valuative Log Space

In this chapter, we define the valuative log space X^{val} associated to a log scheme X and prove the structure sheaf $\mathcal{O}_{X^{val}}$ is coherent if X is a locally Noetherian fine log scheme.

4.1 The Log Space

Definition 4.1.1. A *log space* is a locally ringed space endowed with a log structure.

A log structure $\mathcal{M} \to \mathcal{O}_X$ on a locally ringed space X is called *saturated* (resp. *valuative*) if the stalk $\mathcal{M}_x$ is a saturated (resp. valuative) monoid for all $x \in X$.

Proposition 4.1.2. *([20, Proposition 1.2.9]) Let X be a log scheme whose log structure is quasi-coherent. Then, there exists a scheme X^{sat} endowed with an saturated log structure and with a morphism of log spaces $X^{sat} \to X$ having the following universal property: If Y is a log space whose log structure is saturated, any morphism $Y \to X$ of log spaces factors uniquely through $X^{sat} \to X$.*

Locally on X, X^{sat} is described as follows: Assume the log structure of X is associated to a homomorphism $P \to \mathcal{O}_X$ for a monoid P. Then,

$$X^{sat} = X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{sat}],$$

where P^{sat} is the saturation of P in P^{gp} and it is endowed with the log structure associated to the canonical map to the second factor

$$P^{sat} \to \mathcal{O}_X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{sat}]$$

Theorem 4.1.3. *([20, Theorem 1.3.1]) Let X be a scheme with a quasi-coherent log structure. Then there exists a valuative log space X^{val} with a morphism of log spaces $X^{val} \to X$ having the following universal property: For any valuative log space Y, any morphism $Y \to X$ factors uniquely through $X^{val} \to X$.*

Locally on X, X^{val} may be described as follows: (Keep in mind that we assume X is integral and all monoids are cancellative.) Let $P \to \mathcal{M}_X$ be a chart for $\mathcal{M} \to \mathcal{O}_X$ and let Φ be the set of all finitely generated non-empty proper ideals of P. Endow Φ with the following directed ordering: $J \geq K \Leftrightarrow J = K + K'$ for some finitely generated ideal $K' \subseteq P$,

$$X^{val} = \varprojlim_{K \in \Phi} X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Bl}_K(\operatorname{Spec} \mathbb{Z}[P]).$$

So, by Lemma 3.3.6, if X is a fine log scheme, then $X^{val} \cong (X^{sat})^{val}$.

4.2 The Coherence Theorem

We say a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on a ringed space $(X, \mathcal{O})$ is of *finite type* if locally on X there is a positive integer m, and a surjective homomorphism of $\mathcal{O}$-modules $\mathcal{O}^m \to \mathcal{F}$. We say a sheaf $\mathcal{F}$ is of *relation finite type* if, for every open subset $U \subseteq X$ and every homomorphism of $\mathcal{O}|_U$-modules $\mathcal{O}|_U^m \to \mathcal{F}|_U$, the kernel is of finite type. We say a sheaf $\mathcal{F}$ is of *coherent* if it is both of finite type and relation finite type. The category of coherent $\mathcal{O}$-modules on X is closed under taking direct sums, images, kernels, and cokernels. If a sequence of coherent $\mathcal{O}$-modules on X is exact at $x \in X$, it is exact in a neighborhood of x. The sum and intersection of two coherent subsheaves of a coherent sheaf are coherent. In particular, we will use the following facts:

(1) If $\mathcal{O}$ is coherent and $\mathcal{I} \subseteq \mathcal{O}$ is an ideal of finite type, then $\mathcal{O}/\mathcal{I}$ is a coherent sheaf of $\mathcal{O}$-modules.

(2) Let $i : X \hookrightarrow Y$ be an inclusion of a closed subspace into a topological space Y and let $\mathcal{O}$ be a sheaf of rings on X. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is coherent if and only if $i_*\mathcal{F}$ the trivial extension of $\mathcal{F}$ to Y is a coherent sheaf of $i_*\mathcal{O}$-modules.

For more on coherent sheaves, see Grauert and Remmert [11, Annex §§3-4].

Proposition 4.2.1. *If $X = X(\Delta)_R$ is a toric variety over a Noetherian ring, then the structure sheaf $\mathcal{O}_{X^{val}}$ of X^{val} is universally coherent.*

Proof. Let $f_1, f_2, \dots, f_m$ be sections of $\mathcal{O}_{X^{val}}[t_1, t_2, \dots, t_n]$ on some open $U \subseteq X^{val}$. We must show that the module of relations $\mathrm{Syz}_1(f_1, f_2, \dots, f_m)$ is of finite type. We may assume U is the pre-image of an open set $U_{\Delta'} \subseteq X(\Delta')_R$ for some log blowup Δ' of Δ and the f_is descend to $U_{\Delta'}$. By the proof of part (2) of the flattening theorem applied to $U_{\Delta'} \times \operatorname{Spec} \mathbb{Z}[t_1, t_2, \dots, t_n]$, there is a log blowup Δ'' of Δ' with projection $X(\Delta'')_{R[t_1,t_2,\dots,t_n]} \to X(\Delta')_{R[t_1,t_2,\dots,t_n]}$ such that the strict transform of $\mathrm{Syz}_1(f_1, f_2, \dots, f_m)$ is t-flat. That is, the syzygy module on the pre-image, $U_{\Delta''} \subseteq X(\Delta'')$, of $U_{\Delta'}$ is t-flat. Therefore, by Proposition 3.3.1, for every further log blowup Δ''' of Δ'', the syzygy module on $U_{\Sigma''}$ is t-flat. In particular, the pullback of the syzygy module on $U_{\Sigma''}$ generates the syzygy module on, $U_{\Sigma'''}$, the pre-image of $U_{\Sigma''}$. In particular, $\mathrm{Syz}_1(f_1, f_2, \dots, f_m)$ is of finite type. It is generated by the generators on $U_{\Sigma''}$. □

Theorem 4.2.2. *If X is a locally Noetherian fine log scheme, then the structure sheaf $\mathcal{O}_{X^{val}}$ of X^{val} is universally coherent.*

Proof. The question is local. So, we may assume $X = \operatorname{Spec} R$ is affine and there is a global fine chart $\beta : P \to R$. Using the universal property of X^{sat} and the universal property of X^{val}, we see $X^{val} = (X^{sat})^{val}$. Furthermore, since X is a coherent log scheme, X^{sat} is a fine saturated log scheme. So, by replacing X by X^{sat} if necessary, we may assume X is a fine saturated log scheme and P is a normal affine semigroup as in Corollary 1.3.4. The map $R[P] \to R$ given by $t^p \mapsto \beta(p)$ induces a closed immersion of X into a toric variety, $Y = \operatorname{Spec} R[P]$, over R. Now we have an induced Cartesian square of ringed spaces:

$$\begin{array}{ccc} (X^{val}, \mathcal{O}_{X^{val}}[t_1, t_2, \dots, t_n]) & \xrightarrow{i} & (Y^{val}, \mathcal{O}_{Y^{val}}[t_1, t_2, \dots, t_n]) \\ \downarrow & & \downarrow \\ X & \rightarrow & Y \end{array}$$

$i_* \mathcal{O}_{X^{val}}[t_1, t_2, \dots, t_n]$ is an $\mathcal{O}_{Y^{val}}[t_1, t_2, \dots, t_n]$-module of finite presentation, so it is a coherent $\mathcal{O}_{Y^{val}}[t_1, t_2, \dots, t_n]$-module by Proposition 4.2.1. Since i is a closed immersion given by a coherent sheaf of ideals and $i_* \mathcal{O}_{X^{val}}[t_1, t_2, \dots, t_n]$ is a coherent sheaf of rings, $\mathcal{O}_{X^{val}}[t_1, t_2, \dots, t_n]$ is coherent. □

Chapter 5

Log Regularity

This chapter generalizes much of Kato [23] by relaxing his condition (S) to condition $(*)$ below. By doing so, we work with toric log schemes rather than fine saturated log schemes.

5.1 Definition of Toric Singularity

In this chapter, we will mainly consider log schemes which satisfy the following condition $(*)$.

$(*)$: $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is toric and its underlying scheme X is locally Noetherian.

Warning 5.1.1. It is not enough to ask for a covering by charts given by finitely generated and torsion-free monoids. Let k be a field and let $P = \langle (1,0), (1,1), (0,2) \rangle \subset \mathbb{N}^2$, then P is finitely generated and torsion-free. But, the canonical log structure on $k[P]$ does not satisfy $(*)$. Consider the prime $\mathfrak{p} = (t^{(1,0)}, t^{(1,1)})$, $\overline{\mathcal{M}}_{\operatorname{Spec} k[P], \mathfrak{p}} \cong \langle a, b \mid 2a = 2b \rangle$ is not torsion-free.

Definition 5.1.2. Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a log scheme satisfying condition $(*)$. We say $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is *logarithmically regular* at x, or $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ has (at worst) a *toric singularity* at x, if the following two conditions are satisfied.

(i) $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ is a regular local ring.

(ii) $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})) + \operatorname{rank}(\overline{\mathcal{M}}^{gp}_{X,x})$.

We say $(X, \mathcal{M})$ is *log regular* if $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at each point $x \in X$.

Lemma 5.1.3. [23, Lemma (2.3)] *Let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a log scheme satisfying* $(*)$ *and let $x \in X$. Then*

$$\dim(\mathcal{O}_{X,x}) \le \dim(\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})) + \operatorname{rank}(\overline{\mathcal{M}}^{gp}_{X,x}).$$

Proof. Let R, I, $\mathfrak{m}$, and k equal $\mathcal{O}_{X,x}$, $I(\mathcal{M}^+_{X,x})$, $\mathfrak{m}_x$ and $\mathcal{O}_{X,x}/\mathfrak{m}_x$ respectively. Let $d = \dim(R/I)$ and take $r_1, \ldots, r_d \in \mathfrak{m}$ such that $R/(I, r_1, \ldots, r_d)$ is of finite length. Let $P = \overline{\mathcal{M}}_{X,x}$ and take a section $\varphi : P \to \mathcal{M}_{X,x}$ of $\mathcal{M}_{X,x} \to P$ as in Corollary 1.3.4. Let $\widehat{R}$ be the completion of R. If R contains a field, there exists a section of $\widehat{R} \to k$ by Matsumura [26, Theorem 28.3]. Such a section and the map $P \xrightarrow{\varepsilon\circ\varphi} \mathcal{O}_{X,x}$ induce a finite homomorphism

$$\psi : k[[P]][[t_1, \ldots, t_d]] \to \widehat{R};\ t_i \mapsto r_i.$$

by the proof of [26, Theorem 29.4(iii)]. According to Arnold & Gilmer [1],

$$\dim(k[P][t_1, \ldots, t_d]_{(I(P^+), t_1, \ldots, t_d)}) = d + \operatorname{rank}(P^{gp})$$

thus by [26, Theorem 15.1],

$$\dim(k[[P]][[t_1, \ldots, t_d]]) = d + \operatorname{rank}(P^{gp}).$$

Hence we have

$$\dim(R) \le \dim(k[[P]][[t_1, \ldots, t_d]]) = d + \operatorname{rank}(P^{gp}),$$

which proves the lemma is this case. If R does not contain a field, take a complete discrete valuation ring S with residue field k in which the prime number $p = \operatorname{char}(k)$ is a prime element. By [26, Theorem 29.2], there exists a lifting $S \to \widehat{R}$ of $S \to k$. Thus we obtain a finite S-homomorphism

$$\psi : S[[P]][[t_1, \ldots, t_d]] \to \widehat{R};\ t_i \mapsto r_i.$$

We will show that ψ is not injective. We will then have

$$\dim(R) \le \dim(S[[P]][[t_1, \ldots, t_d]]) - 1 = d + \operatorname{rank}(P^{gp}),$$

which will finish the proof of this lemma. Now if ψ were injective, there exists a prime ideal $\mathfrak{p}$ of $\widehat{R}$ such that $\psi^{-1}(\mathfrak{p}) = (I(P^+), t_1, \ldots, t_d)$ by the lying-over part of the going-up theorem. Then we would have $(I, r_1, \ldots, r_d) \subset \mathfrak{p}$ and $\mathfrak{p} = \mathfrak{m}$. Thus we would have $\psi^{-1}(\mathfrak{p}) = (\mathfrak{m}_S, I(P^+), t_1, \ldots, t_d)$, contradicting the choice of $\mathfrak{p}$. $\square$

5.2 Completed Toric Singularities

In this section, let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a log scheme satisfying $(*)$.

Lemma 5.2.1. [23, Lemma (3.5)] *Let R be a ring, let π be a nonzero-divisor of R, let P and Q be affine semigroups, and let $P \to Q$ be an injective homomorphism. Let θ be an element of $R[[P]]$ such that $\theta \equiv \pi \bmod I(P^+)$. Then $R[[P]]/(\theta) \to R[[Q]]/(\theta)$ is injective.*

Proof. Assume

$$\left(\sum_{q\in Q} r_q t^q\right)\theta = \sum_{p\in P} r_p t^p, \qquad r_p, r_q \in R \tag{5.1}$$

in $R[[Q]]$. We wish to show that $\sum_{p\in P} r_p t^p$ is a multiple of θ in $R[[P]]$. Consider the set

$$\Phi = \{q \in Q \setminus P \mid r_q \neq 0\},$$

if Φ is empty, there is nothing to do, so assume Φ is nonempty, and let (t^{q_0}) be maximal in the set of ideals of $\mathbb{Z}[Q]$ of the form (t^q) such that $q \in \Phi$. Write $\theta = \pi + \sum_{p\neq 0} r'_p t^p$, then, by looking at the coefficient of t^{q_0} in (5.1), we see

$$0 = \pi r_{q_0} + \sum_{\substack{(p,q)\in P\times Q \\ p+q=q_0, p\neq 0}} r'_p r_q.$$

Since $\pi r_{q_0} \neq 0$, we have $r_q \neq 0$ for some $q \in Q$ such that there exists $p \in P, p \neq 0$ satisfying $p + q = q_0$. Thus $q \in \Phi$ and $(t^{q_0}) \subset (t^q)$, contradicting the maximality of (t^{q_0}). □

Lemma 5.2.2. [23, Lemma (3.4)] *Let R be a ring, let π be a nonzero-divisor of R such that $R/(\pi)$ is an integral domain, let P be an affine semigroup. Let θ be an element of $R[[P]]$ such that $\theta \equiv \pi \bmod I(P^+)$. Then $R[[P]]/(\theta)$ is an integral domain.*

Proof. Take an injective homomorphism $P \hookrightarrow \mathbb{N}^d$ for some $d \geq 0$. Because

$$\mathrm{gr}_{((\mathbb{N}^d)^+)}(R[[\mathbb{N}^d]]/(\theta)) \cong (R/\pi R)[\mathbb{N}^d]$$

is an integral domain, $R[[\mathbb{N}^d]]/(\theta)$ is an integral domain. Hence, it suffices to show that $R[[P]]/(\theta) \to R[[\mathbb{N}^d]]/(\theta)$ is injective. Hence we are reduced to the Lemma 5.2.1. □

Theorem 5.2.3. [23, Theorem (3.2)] *Let $x \in X$. Assume $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ is regular, let $P = \overline{\mathcal{M}}_{X,x}$, let φ be a section of $\mathcal{M}_{X,x} \to P$ as in Corollary 1.3.4, and let $r_1, \ldots, r_d \in \mathcal{O}_{X,x}$ such that $(r_i \bmod I(x, \mathcal{M}))_{1\leq i\leq d}$ is a regular system of parameters of $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$.*

1. *If $\mathcal{O}_{X,x}$ contains a field, let k be a subfield of $\widehat{\mathcal{O}}_{X,x}$ such that $\widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x \cong k$. Then $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at x if and only if the surjective homomorphism*
$$\psi : k[[P]][[t_1, \dots, t_d]] \to \widehat{\mathcal{O}}_{X,x};\ t_i \mapsto r_i$$
is an isomorphism.

2. *If $\mathcal{O}_{X,x}$ does not contain a field, let R be a complete discrete valuation ring in which $p = \mathrm{char}(\mathcal{O}_{X,x}/\mathfrak{m}_x)$ is a prime element and fix a homomorphism $R \to \widehat{\mathcal{O}}_{X,x}$ which induces $R/pR \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x$. Then $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at x if and only if the kernel of the surjective homomorphism*
$$\psi : R[[P]][[t_1, \dots, t_d]] \to \widehat{\mathcal{O}}_{X,x};\ t_i \mapsto r_i$$
is generated by an element θ such that
$$\theta \equiv p \bmod (I(P^+), t_1, \dots, t_d)$$

Proof. The "if" directions of the theorem are evident. We will prove the "only if" directions. Assume first that $\mathcal{O}_{X,x}$ contains a field and let $k \to \widehat{\mathcal{O}}_{X,x}$ be as above. Now since
$$\dim(k[[P]][[t_1, \dots, t_d]]) = \mathrm{rank}(P^{gp}) + d = \dim(\mathcal{O}_{X,x}) = \dim(\widehat{\mathcal{O}}_{X,x}),$$
and since $k[[P]][[t_1, \dots, t_d]]$ is an integral domain, the surjective map ψ in (1) is an isomorphism.

Next assume that $\mathcal{O}_{X,x}$ does not contain a field, and let $R \to \widehat{\mathcal{O}}_{X,}$ be as in (2). Since the map ψ sends the ideal $(I(P^+), t_1, \dots, t_d)$ of $R[[P]][[t_1, \dots, t_d]]$ onto $\widehat{\mathfrak{m}}$, there exists $\theta \in \ker(\psi)$ such that
$$\theta \equiv p \bmod (I(P^+), t_1, \dots, t_d).$$
The Lemma 5.2.2 shows that $R[[P]][[t_1, \dots, t_d]]/(\theta)$ is an integral domain, once again by comparing dimensions we see that ψ is an isomorphism. □

Corollary 5.2.4. [23, Theorem (3.1)]

1. *$(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at x if and only if there exists a complete regular local ring R, an affine semigroup P, and an isomorphism*
$$R[[P]]/(\theta) \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}$$
with $\theta \in R[[P]]$ satisfying the following conditions (i) and (ii).

(i) *The constant term of* θ *belongs to* $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$.

(ii) *The inverse image of* $\mathcal{M}$ *on* $\operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$ *is induced by the map* $P \to \widehat{\mathcal{O}}_{X,x}$.

2. *Assume* $\mathcal{O}_{X,x}$ *contains a field. Then,* $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ *is log regular at* x *if and only if there exists a field* k*, an affine semigroup* P*, and an isomorphism*

$$k[[P]][[t_1, \ldots, t_d]] \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}$$

for some $d \geq 0$ *satisfying the condition (ii) in (1).*

Proposition 5.2.5. $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ *is log regular at* x *if and only if* $\mathcal{O}_{X,x}$ *is t-flat and* $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ *is a regular local ring.*

Proof. Since $\widehat{\mathcal{O}}_{X,x}$ is faithfully flat over $\mathcal{O}_{X,x}$, $\mathcal{O}_{X,x}$ is t-flat if and only if $\widehat{\mathcal{O}}_{X,x}$ is t-flat. By Theorem 5.2.3, $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at x if and only if $\operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$ is log regular at x. So, we may assume $\mathcal{O}_{X,x} = \widehat{\mathcal{O}}_{X,x}$. If $\mathcal{O}_{X,x}$ is t-flat, then the equivalence of (1) and (4') in the Theorem 2.2.2 shows $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})) + \operatorname{rank}(\overline{\mathcal{M}}^{gp}_{X,x})$. On the other hand, if $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at x, then Theorem 5.2.3 and the equivalence of (1) and (4') in the Theorem 2.2.2 show $\mathcal{O}_{X,x}$ is t-flat. $\square$

Theorem 5.2.6. [23, Theorem (6.2)] *Let* P *be an affine semigroup, let* $P = \overline{\mathcal{M}}_{X,x}$ *and take a section* $\varphi : P \to \mathcal{M}_{X,x}$ *of* $\mathcal{M}_{X,x} \to P$ *as in Corollary 1.3.4. Assume* $\mathcal{O}_{X,x}$ *contains a field* k*. Then* $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ *is log regular at* x *if and only if* $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ *is a regular local ring and the map* $k[P] \to \mathcal{O}_{X,x}$ *induced by* φ *is flat.*

Proof. This follows immediately from Theorem 5.2.5, Theorem 2.2.2, and the local criterion for flatness (see EGA III [12, 0_{III} (10.2.2)]). $\square$

5.3 Some Properties of Toric Singularities

Kato reminds us that if $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is a log regular scheme satisfying his condition (S) then its underlying scheme is Cohen-Macaulay and normal, see Hochster [13]. Let k be a field. If the canonical log structure on $\operatorname{Spec} k[P]$ satisfies $(*)$, it need not be Cohen-Macaulay nor normal. Consider the monoids $\langle (4,0), (3,1), (1,3), (0,4) \rangle \subset \mathbb{N}^2$ and $\langle 2, 3 \rangle \subset \mathbb{N}$. Information on when affine semigroup rings are Cohen-Macaulay and its dependence on the characteristic, can be found in [41, 42]. Information on the local cohomology modules and dualizing complexes of affine semigroup rings is found in [18] and [36].

Theorem 5.3.1. [23, Theorem (4.2)] *Let* $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ *and* $(Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y)$ *be log regular schemes. Let* $f : (X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X) \to (Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y)$ *be a morphism whose underlying map of schemes is a closed immersion and assume* $f^*\mathcal{M}_Y \cong \mathcal{M}_X$*. Then the underlying map of schemes of* f *is a regular immersion.*

Proof. Let $x \in X$, and let J be the kernel of $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Since $f^*\mathcal{M}_Y \cong \mathcal{M}_X$, we have

$$\mathcal{O}_{Y,f(x)}/(J + I(\mathcal{M}^+_{Y,f(x)})) \xrightarrow{\cong} \mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x}).$$

Since $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ and $\mathcal{O}_{Y,f(x)}/I(\mathcal{M}^+_{Y,f(x)})$ are regular, we see that there exist

$$r_1, \ldots, r_d \in J$$

whose images in $\mathcal{O}_{Y,f(x)}/I(\mathcal{M}^+_{Y,f(x)})$ form part of a regular system of parameters and generate the kernel of

$$\mathcal{O}_{Y,f(x)}/I(\mathcal{M}^+_{Y,f(x)}) \to \mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x}).$$

By Theorem 5.2.3, $r_1, \ldots, r_d$ form a regular sequence of $\mathcal{O}_{Y,f(x)}$. □

5.4 Localization

For the duration of this section, let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a log scheme satisfying condition $(*)$.

Theorem 5.4.1. [23, Proposition (7.2)] *Let* $x \in X$ *and suppose* $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ *is log regular at* x*. Let* $\mathfrak{p}$ *be a prime ideal of* $\mathcal{M}_{X,x}$ *and endow* $X' = \mathrm{Spec}(\mathcal{O}_{X,x}/I(\mathfrak{p}))$ *with the log structure associated to* $\mathcal{M}_{X,x} \setminus \mathfrak{p} \to \mathcal{O}_{X,x}/I(\mathfrak{p})$*.* $(X', \alpha_{X'} : \mathcal{M}_{X'} \to \mathcal{O}_{X'})$ *is log regular at* $x \in X'$*.*

Proof. Let $P = \overline{\mathcal{M}}_{X,x}$ and take a section $\varphi : P \to \mathcal{M}_{X,x}$ of $\mathcal{M}_{X,x} \to P$ as in Corollary 1.3.4. By Theorem 5.2.3, φ extends to an isomorphism $R[[P]]/(\theta) \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}$ for some complete local ring R and an element θ of $R[[P]]$ whose constant term belongs to $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Let $\overline{\mathfrak{p}} = \varphi^{-1}(\mathfrak{p})$ be the prime ideal of P corresponding to $\mathfrak{p}$. Then, $\widehat{\mathcal{O}}_{X,x}/I(\mathfrak{p}) = \widehat{\mathcal{O}}_{X,x}/I(\overline{\mathfrak{p}})$ is isomorphic to $R[[P \setminus \overline{\mathfrak{p}}]]/(\overline{\theta})$ where $\overline{\theta} \equiv \theta \bmod I(\mathfrak{p})$. Note that the log structure $\mathcal{M}_{X'}$ is associated to $P \setminus \overline{\mathfrak{p}} \to \mathcal{O}_{X,x}$. By Corollary 5.2.4, we are done. □

Corollary 5.4.2. [23, Corollary (7.3)] *With the notation as in the previous theorem,* $I(\mathfrak{p})$ *is a prime ideal of height* $\dim(\mathcal{M}_{X,x})_{\mathfrak{p}}$*.*

Lemma 5.4.3. *Let R be a local ring, let P be a 1-dimensional affine semigroup, let $\beta : P \to R$ be a monoid homomorphism with respect to multiplication, and let $\mathfrak{p} \subset R$ be a prime ideal such that $\beta^{-1}(\mathfrak{p}) = \emptyset$. If* $\operatorname{Spec}(P \xrightarrow{\beta} R)$ *is log regular, then $R_{\mathfrak{p}}$ is a regular local ring.*

Proof. By Theorem 5.2.3 $\widehat{R} = S[[P]]/(\theta)$ for some complete regular local ring S and some θ whose constant term is contained in $\mathfrak{m}_S \setminus \mathfrak{m}_S^2$. Note that $\mathbb{N}$ is the saturation of P and there exists an $n \in \mathbb{N}$ such that $m \in P$ whenever $m \geq n$. Let F be the image of P in R and consider the following diagram:

$$\begin{array}{ccccc} R & \to & S[[P]]/(\theta) & \to & S[[\mathbb{N}]]/(\theta) \\ \downarrow & & \downarrow & & \downarrow \\ R_F & \to & (S[[P]]/(\theta))_F & = & S((\mathbb{N}))/(\theta) \\ \downarrow & & \downarrow & & \\ R_{\mathfrak{p}} & \to & (R \setminus \mathfrak{p})^{-1} S[[P]]/(\theta) & & \end{array}$$

$R_{\mathfrak{p}}$ is faithfully flat under $(R \setminus \mathfrak{p})^{-1} S[[P]]/(\theta)$ and the latter is regular since it is a localization of $S((\mathbb{N}))/(\theta)$. Hence $R_{\mathfrak{p}}$ is regular by faithfully flat descent, see Matsumura [26, Theorem 23.7]. □

Theorem 5.4.4. [23, Proposition (7.1)] *Let $x \in X$, and assume that $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at x. Then for any $y \in X$ such that $x \in \overline{\{y\}}$, $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at y.*

Proof. Let $P = \overline{\mathcal{M}}_{X,x}$ and take a homomorphism $\varphi : P \to \mathcal{O}_{X,x}$ which induces $\mathcal{M}_X$ at x. Let $y \in X$, $x \in \overline{\{y\}}$, and let $\mathfrak{p}$ be the inverse image in P of the prime ideal of $\mathcal{O}_{X,x}$ corresponding to y. We shall use induction on $\dim(P \setminus \mathfrak{p})$.

By Proposition 5.2.5, $\mathcal{O}_{X,x}$ is t-flat (over $\mathbb{Z}[P]$). So, $\mathcal{O}_{X,y}$ is t-flat (over $\mathbb{Z}[P_{\mathfrak{p}}]$). Therefore, it suffices to show that $\mathcal{O}_{X,y}/I(\mathcal{M}_{X,y}^+)$ is a regular local ring. Recall that $I(\mathfrak{p})$ is a prime ideal of $\mathcal{O}_{X,x}$ by Corollary 5.4.2, and that $I(\mathcal{M}_{X,y}^+) = I(\mathfrak{p})$. We may assume $\mathfrak{p} \neq \varphi^{-1}(\mathfrak{m}_x)$, for if $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_x)$, then $\mathcal{O}_{X,y}/I(\mathcal{M}_{X,y}^+)$ is a localization of $\mathcal{O}_{X,x}/I(\mathcal{M}_{X,x}^+)$ and we are done. Take a prime $\mathfrak{q} \subset P$ such that $\mathfrak{p} \subset \mathfrak{q} \subseteq \varphi^{-1}(\mathfrak{m}_x)$ and $\dim(P \setminus \mathfrak{q}) = \dim(P \setminus \mathfrak{p}) - 1$. Let $z \in \operatorname{Spec}(\mathcal{O}_{X,x})$ be the prime ideal $I(\mathfrak{q})$. By induction $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at z. Hence by Theorem 5.4.1, the log scheme $\operatorname{Spec}(\mathcal{O}_{X,z}/I(\mathfrak{p}))$ endowed with the log structure associated to $P_{\mathfrak{q}} \setminus (\mathfrak{p} + P_{\mathfrak{q}}) \to \mathcal{O}_{X,z}/I(\mathfrak{p})$ is log regular at z. Since $\dim(P_{\mathfrak{q}} \setminus (\mathfrak{p} + P_{\mathfrak{q}})) = 1$, by Lemma 5.4.3 we are done. □

5.5 Log Smooth Morphisms

Let $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ and $(Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y)$ be log schemes satisfying $(*)$, and let $f : (X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X) \to (Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y)$ be a morphism. Then, the following two conditions (i) and (ii) are equivalent. We say f is log smooth if f satisfies these conditions. This equivalence was shown in Kato [21] for log structures on étale sites, and the proof there works for the present situation (log structures on Zariski sites). See Kato [20, Section 3.1.5]. So, we omit the proof here.

(i) Assume we are given a commutative diagram of log schemes

$$\begin{array}{ccc} (T, \alpha_T : \mathcal{M}_T \to \mathcal{O}_T) & \xrightarrow{g} & (X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X) \\ i \downarrow & & \downarrow f \\ (T', \alpha_{T'} : \mathcal{M}_{T'} \to \mathcal{O}_{T'}) & \xrightarrow[g']{} & (Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y) \end{array}$$

such that $(T, \alpha_T : \mathcal{M}_T \to \mathcal{O}_T)$ and $(T', \alpha_{T'} : \mathcal{M}_{T'} \to \mathcal{O}_{T'})$ satisfy $(*)$, the morphism of schemes $i : T \to T'$ is a closed immersion, T is defined in T' by a nilpotent ideal of $\mathcal{O}_{T'}$ and $i^*\mathcal{M}_{T'} \to \mathcal{M}_T$ is an isomorphism. Then, locally on T' there is a morphism $h : (T', \alpha_{T'} : \mathcal{M}_{T'} \to \mathcal{O}_{T'}) \to (X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ such that $h \circ i = g$ and $f \circ h = g'$. Furthermore, the underlying morphism of schemes $X \to Y$ is locally of finite type.

(ii) Étale locally on X and Y, there exist finitely generated, torsion-free monoids P and Q, an injective homomorphism $h : P \to Q$ such that the order of the torsion part of $Q^{gp}/h^{gp}(P^{gp})$ is invertible on X, and a commutative diagram of log schemes of the form

$$\begin{array}{ccc} (X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X) & \to & \mathrm{Spec}(Q \hookrightarrow \mathbb{Z}[Q]) \\ f \downarrow & & \downarrow \text{induced by } h \\ (Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y) & \to & \mathrm{Spec}(P \hookrightarrow \mathbb{Z}[P]) \end{array}$$

such that the inverse image of P on Y is $\mathcal{M}_Y$, the inverse image of Q on X is $\mathcal{M}_X$ and the induced morphism of the underlying schemes

$$X \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\,\mathbb{Z}[Q]$$

is smooth (in the classical sense).

Theorem 5.5.1. [23, Theorem (8.2)] *Let $f : (X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X) \to (Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y)$ be a log smooth morphism between log schemes satisfying $(*)$, and assume $(Y, \alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y)$ is log regular. Then $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular.*

Proof. We may work étale locally, so we may assume that: there exist finitely generated torsion-free monoids P and Q, an injective homomorphism $h : P \to Q$ such that the order of the torsion part of $Q^{gp}/h^{gp}(P^{gp})$ is invertible on X, P induces $\mathcal{M}_Y$, Q induces $\mathcal{M}_X$, and $X = Y \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$. Let $x \in X$ and $y = f(x) \in Y$. To prove that $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular at y, we may further assume that the inverse image of $\mathcal{O}^*_{Y,y}$ in P is P^* and the inverse image of $\mathcal{O}^*_{X,x}$ in Q is Q^*. Assuming this, I claim $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ is a local ring of a smooth algebra (in the classical sense) over $\mathcal{O}_{Y,y}/I(\mathcal{M}^+_{Y,y})$.

Indeed $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ is a local ring of

$$\begin{aligned}(\mathbb{Z}[Q]/I(Q^+)) \otimes_{\mathbb{Z}[P]} \mathcal{O}_{Y,y} &\cong (\mathbb{Z}[Q]/I(Q^+)) \otimes_{\mathbb{Z}[P]/I(P^+)} \mathcal{O}_{Y,y}/I(\mathcal{M}^+_{Y,y}) \\ &\cong (\mathbb{Z}[Q^*]) \otimes_{\mathbb{Z}[P^*]} \mathcal{O}_{Y,y}/I(\mathcal{M}^+_{Y,y}).\end{aligned}$$

It is enough to notice that $\mathbb{Z}[\frac{1}{n}][P^*] \to \mathbb{Z}[\frac{1}{n}][Q^*]$ is smooth with n the order of the torsion part of $Q^*/h(P^*)$, and that the torsion part of $Q^*/h(P^*)$ injects into the torsion part of $Q^{gp}/h^{gp}(P^{gp})$.

Since the ring $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ is regular, by Proposition 5.2.5 it suffices to prove that $\mathcal{O}_{X,x}$ is t-flat. This follows from Proposition 2.2.4. $\square$

Theorem 5.5.2. [23, Proposition (8.3)] *Let k be a field and let $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ be a log scheme satisfying $(*)$ such that the underlying scheme X is a k-scheme which is locally of finite type. Then:*

1. *If $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ is log smooth over* $\operatorname{Spec}(\{0\} \to k)$*, then $(X, \alpha_X : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular.*

2. *The converse of (1) is true if k is perfect.*

Proof. (1) follows from Theorem 5.5.1.

Assume k is perfect and $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is log regular. Let $x \in X$, and take an affine semigroup P, and a homomorphism $\varphi : P \to \mathcal{O}_{X,x}$ which induces $\mathcal{M}_X$ at x. It suffices to show that $\mathcal{O}_{X,x}$ is a local ring of a smooth algebra over $k[P]$. But, the map $k[P] \to \mathcal{O}_{X,x}$ is flat by Theorem 5.2.6 and its "fiber"

$$k = k[P]/I(P^+) \to \mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$$

is smooth (for k is perfect and $\mathcal{O}_{X,x}/I(\mathcal{M}^+_{X,x})$ is regular). $\square$

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