Log geometry and multiplier ideals

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February 28, 2011

Overview

I work in combinatorics, algebraic geometry, convex geometry and commutative algebra while staying informed on certain topics in category theory and ring theory. In particular, I focus on toric varieties and singularity theory. The study of toric varieties lies at the intersection of combinatorics, algebraic geometry, convex geometry and integer programming. There is a correspondence between certain combinatorial objects in convex geometry, cones and fans, and the geometry of toric varieties. For example, there are techniques based on toric geometry for counting the lattice points in a lattice polytope. Because of this correspondence between combinatorial objects in convex geometry and toric geometry, objects in toric geometry tend to be more concrete and computable than they are in algebraic geometry in general. At the same time, toric varieties provide a large enough class of geometric objects to test many conjectures. For example, the class of toric varieties includes products of projective spaces, many spaces with mild singularities, and some compact nonprojective varieties.

The jumping numbers of a singular variety Y embedded in a smooth complex variety X form an interesting invariant of the pair (X, Y). The jumping numbers are a sequence of positive rational numbers computed from— and reflecting subtle information about— an embedded resolution of singularities of the pair. For example, in the simplest case where Y is a smooth hypersurface in X, the jumping numbers are simply the positive integers. But the sequence of jumping numbers becomes increasingly complicated as a resolution of singularities requires more blowings up or as functions vanishing on Y vanish to higher orders along the resulting exceptional divisors. Jumping numbers, also known as jumping coefficients, were first explicitly defined in [4] as those numbers λ for which the multiplier ideal of the pair $(X, \lambda Y)$ makes a discrete "jump", though these natural invariants arose earlier in several contexts; see [9], [10], and [19].

This research statement starts with my study of toric geometry. The first four sections start by talking about my ideas about the classic theory and move toward logarithmic geometry of schemes that behave like toric varieties, sections 5 and 6, explain my work on multiplier ideals, and the last section lists some my future plans. In Section 1, I describe an invariant of a (classical) toric variety that measures its failure to be smooth and generalizes the notion of the

multiplicity of a simplicial cone. My invariant gives us more insight into the resolution of singularities on toric varieties. In Section 2, I describe a combinatorial gadget that replaces the fans of classical toric geometry. My more general notion of a fan allows one to apply toric techniques to non-normal toric varieties that are neither affine nor projective. In Section 3, I describe a few results about local rings whose completions are isomorphic to the completions of local rings of toric varieties and mixed characteristic analogs of these rings. It is my not necessarily normal version of some work of Kato on certain normal rings. In Section 4, I describe a flattening theorem that is adapted to the toric setting.

1 Resolution of singularities on (classic) toric varieties

This section is about the contents of [15].

Let $X = X(\Delta)$ be a toric variety. Every Weil divisor on X is $\mathbb{Q}$-Cartier only if X is simplicial. Furthermore, every Weil divisor on X is Cartier only if X is smooth. In fact, the group of Weil divisors modulo Cartier divisors, $G = \mathbb{Z}^{\Delta(1)}/\operatorname{Div}_T(X)$, generalizes the notion of the multiplicity of a simplicial cone. If Δ is the set of faces of a simplicial cone σ, then the multiplicity of σ is the order of G.

Suppose D is a prime invariant Weil divisor on X that is not $\mathbb{Q}$-Cartier. For each cone σ such that the 1-cone representing D is an edge of σ, let n_σ be the least common multiple of the orders of the minimal generators of the cone $\sigma^\vee$, viewed as elements of $k[S_\sigma]$, at the height one prime of $k[S_\sigma]$ corresponding to D. If the 1-cone representing D is not an edge of σ, let $n_\sigma = 1$. Let n be the least common multiple of the n_σ, then blowing up the ideal associated to nD does not introduce new invariant prime Weil divisors. That is, by repeatedly doing this, we can find a toric blowup $\pi : \widetilde{X} \to X$ such that the induced map on invariant Weil divisors is an isomorphism and $\widetilde{X}$ is simplicial. This blowup corresponds to stellar subdivision of Δ along the 1-cone associated to D.

In particular, stellar subdivision of Δ along the 1-cone associated to a prime invariant Weil divisor that is not $\mathbb{Q}$-Cartier seems to be a preferred method for simplicialization. The resultant variety agrees with the original variety in codimension one and the associated map of varieties is particularly easy to understand. I hope that the study of the group, G, might lead to a notion of "minimal" resolutions of toric singularities and a deeper understanding of the resolution process.

Furthermore, the blowup of an affine toric variety along a integrally closed invariant ideal $I \subseteq k[S]$ can be identified with a toric variety $\operatorname{Spec} k[S]$ we associate with the convex hull of the set of lattice points in $I \cap S$.

This point of view is explained in [15]. Even in the classical setting, this point of view is useful. In particular, from this point of view we can identify the convex geometric objects that appear in Howald [5]. And, we can see that the arguments that appear there aren't limited to the smooth case (see Blickle [1]).

2 Toric schemes

This section reviews the contents of [16].

Consider the set $\{S_\sigma \mid \sigma \in \Delta\}$, where Δ is a (classic) fan. Since each σ is a strongly convex rational polyhedral cone, each S_σ is a finitely generated saturated submonoid of M such that $S_\sigma^{gp} = M$. Since every face of a cone in Δ is also in Δ, the set $\{S_\sigma \mid \sigma \in \Delta\}$ is closed under localization. Furthermore, since the intersection of any two cones in Δ is a face of each, the sum of any two monoids in $\{S_\sigma \mid \sigma \in \Delta\}$ is a localization of each.

In [16], I note that we may construct non-normal toric schemes simply by dropping the saturated condition. That is, by starting with a finite set Σ of finitely generated cancellative torsion-free submonoids of some fixed finitely generated free Abelian group M such that for each $P \in \Sigma$, $P^{gp} = M$; Σ is closed under localization; and $\forall P, Q \in \Sigma$, $P + Q$ is a localization of both P and Q. In an abuse of language, I call such a set Σ a fan. Given a monoid P in Σ, we associate to P the affine scheme $X(P) = \operatorname{Spec} \mathbb{Z}[P]$. Since, for every pair $P, Q \in \Sigma$, $X(P + Q)$ is an open affine subscheme of both $X(P)$ and $X(Q)$, we may obtain a scheme $X(\Sigma)$ from these schemes by gluing them together as in the classical case. Of course, if we are given a base scheme S and a fan Σ, we may form $X(\Sigma) \times S$ the toric scheme over S with fan Σ.

Following DeMeyer, Ford & Miranda [3], we make our fan Δ into a topological space by letting the open sets in the topology be the subsets of Δ that are themselves fans. I noticed that the presheaf that associates to a subfan $\Delta \subseteq \Sigma$, the monoid $\bigcap_{P \in \Delta} P$ is a sheaf, and furthermore, if Σ is the set of localizations of some monoid P, then Σ equipped with its sheaf of monoids is isomorphic to $\operatorname{Spec} P$. So, fans are much like schemes. Of course, we also have a map of monoided spaces $X(\Delta) \to \Delta$ which is a weak algebraic analog of a moment map.

My notion of fan is similar to, but not the same as the notion Kato presented in [7].

3 Toric singularities

This section is about [17] where I remove the normality requirement from several of the statements in [7].

A monoid is said to be *cancellative* if whenever a, b and c are elements such that $a + c = b + c$, it is necessarily the case that $a = b$. A monoid is said to be *torsion-free* if whenever a and b are elements and n is a positive integer such that $na = nb$, it must be the case that $a = b$. A cancellative monoid P is said to be *saturated* if whenever a is an element of the group envelop P^{gp} of P and n is a positive integer such that na is an element of P, it is necessarily the case that a is an element of P. A monoid is said to be *sharp* if its unit group is trivial. We write $P^+ = P \setminus P^*$ for the unique maximal ideal of P.

A log scheme $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is a scheme $(X, \mathcal{O}_X)$ equipped with a homomorphism of sheaves of commutative monoids α with respect to the mul-

tiplication on $\mathcal{O}_X$ such that the restriction $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. We identify $\mathcal{M}_X^*$ with $\mathcal{O}_X^*$ and write $\overline{\mathcal{M}}_X$ for $\mathcal{M}_X/\mathcal{O}_X^*$. A log scheme is said to be *fine* if locally there exists a finitely generated cancellative monoid P and a homomorphism of sheaves of monoids β from the constant sheaf P_X with stalks isomorphic to P to $\mathcal{O}_X$ such that $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ is the pushout of the diagram

$$\begin{array}{c} \beta^{-1}(\mathcal{O}_X^*) \to P_X \\ \downarrow \\ \mathcal{O}_X^* \end{array}$$

induced by the inclusion $\mathcal{O}_X^* \subseteq \mathcal{O}_X$ and β. In this situation, we say $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ is the log structure associated to the pre-log structure β. We say $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is saturated if $\mathcal{M}_X$ is a sheaf of saturated monoids. If $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is a fine (saturated) log scheme, then $\overline{\mathcal{M}}_X$ is a sheaf of finitely generated cancellative sharp (saturated) monoids. Furthermore, if $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is a fine saturated log scheme, then $\overline{\mathcal{M}}_X$ is a sheaf of torsion-free monoids.

Of course, any toric scheme (over $\mathbb{Z}$) has a canonical log structure, namely the log structure associated to the pre-log structure $P_{\operatorname{Spec}\mathbb{Z}[P]} \to \mathcal{O}_{\operatorname{Spec}\mathbb{Z}[P]}$ induced by the canonical inclusion $P \hookrightarrow \mathbb{Z}[P]$ on each affine open subset $\operatorname{Spec}\mathbb{Z}[P]$.

Kato [7] studied the local rings at points of locally Noetherian fine saturated log schemes. Kato proved that on such a log scheme, locally there exists a homomorphism of sheaves of monoids $\beta : (\overline{\mathcal{M}}_{X,x})_X \to \mathcal{O}_X$ such that $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ is the log structure associated to the pre-log structure β. In [17], I remove the normality hypothesis from this theorem. I proved that if $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ is a locally Noetherian fine log scheme such that $\overline{\mathcal{M}}_{X,x}$ is torsion-free, then there is an open neighborhood U of x and a homomorphism of sheaves of monoids $\beta : (\overline{\mathcal{M}}_{X,x})_U \to \mathcal{O}_U$ such that $\alpha|_U$ is the log structure associated to the pre-log structure β.

We will call a local ring $(R, \mathfrak{m})$ equipped with a monoid homomorphism $\beta : P \to R$ to the multiplicative monoid of R such that $\beta^{-1}(R^*) = P^*$ a *log local ring*. If, in addition, P is a finitely generated cancellative monoid and R is Noetherian, we will say $\beta : P \to R$ is a fine Noetherian log local ring. If P/P^* is torsion-free, we can and will assume P is sharp. We will say our log local ring is saturated (resp. torsion-free) if P is saturated (resp. torsion-free). Kato called a saturated fine Noetherian log local ring such that $\overline{R} = R/\beta(P^+)R$ is a regular local ring and $\dim(R) = \dim(\overline{R}) + \operatorname{rank} P^{gp}$ *log regular* and said that it had (at worst) a toric singularity at $\mathfrak{m}$. Such an R has very similar behavior to $\overline{R}[P]$. In particular, Kato proved that there is a complete regular local ring R' such that $\widehat{R} \cong R'[[P]]$, the localization of a log regular local ring at a prime ideal is log regular, and the induced log local ring $P \setminus \mathfrak{p} \to R/\beta(\mathfrak{p})R$ is log regular if $\mathfrak{p}$ is a prime ideal of P.

Suppose we were to consider a torsion-free fine Noetherian log local ring such that $\overline{R} = R/\beta(P^+)R$ is a regular local ring and $\dim(R) = \dim(\overline{R}) + \operatorname{rank} P^{gp}$ to be log regular. With this broadened definition, I proved that there is a complete regular local ring R' such that $\widehat{R} \cong R'[[P]]$, the localization of a log

regular local ring at a prime ideal is log regular, and the induced log local ring $P \setminus \mathfrak{p} \to R/\beta(\mathfrak{p})R$ is log regular if $\mathfrak{p}$ is a prime ideal of P. For the most part, the arguments follow those of Kato. But, one needs a few technical lemmas to get started in the non-normal case.

4 Toric flattening

This section covers other topics in my dissertation [14].

Given a fine log scheme $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$, we define a log blowup of $(X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$ locally as follows: Suppose the log structure on X is given by a finitely generated cancellative monoid P and a pre-log structure $\beta : P_X \to \mathcal{O}_X$, let K be an ideal of P, and let I be the ideal of $\mathbb{Z}[P]$ generated by the image of K. The *log blowup* is given by

$$\mathrm{Bl}_K X = X \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Proj}\left(\bigoplus_{n \geq 0} I^n\right).$$

A log blowup of a toric scheme is a toric morphism. So, we also refer to such maps as toric blowups. Kato's *valuative log space* X^{val} (defined in [6]) is the limit of these blowups in the category of locally ringed spaces equipped with a log structure. We say a sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$ is a sheaf of *log ideals* if it is generated by the image of a sheaf of ideals $\mathcal{K}_X \subseteq \mathcal{M}_X$.

Let $X = X(\Delta)$ be a toric variety and let $\mathcal{F}$ be a coherent sheaf on X. We say $\mathcal{F}$ is t-flat if it is flat at each prime log ideal. With help from O. Gabber, I have proved: (1) t-Flatness is an open condition; (2) there is a local criterion for t-flatness; (3) if $\mathcal{F}$ is t-flat and $\pi : \widetilde{X} \to X$ is a toric blowup, then $\pi^*\mathcal{F}$ is the strict transform of $\mathcal{F}$ and it is t-flat; and (4) there exists a toric blowup $\pi : \widetilde{X} \to X$ such that the strict transform of $\mathcal{F}$ is t-flat. Consequently, the structure sheaf of X^{val} is universally coherent in the toric case despite the fact that log blowups are not flat. In general, the question of whether the structure sheaf of X^{val} is coherent reduces to the toric case. There one uses a careful Noetherian induction. So, the structure sheaf of X^{val} is universally coherent in general.

5 Jumping numbers of plane curves

The jumping numbers of a pair (X, Y) are determined by the exceptional divisors (or valuations) appearing in a log resolution. On the other hand, some exceptional divisors never contribute to the jumping numbers. Since the jumping numbers do not depend on the choice of the log resolution, any divisor obtained by performing an "extraneous blowup" will be irrelevant from the point of view of jumping numbers. But examples show that even some "essential" exceptional divisors do not contribute to the sequence of jumping numbers. What is special about these divisors?

In [12], Karen Smith and I investigate this phenomenon, focusing on curves on a smooth surface, and establish precisely which exceptional divisors in a minimal log resolution of a singular curve on a smooth complex surface are "irrelevant" from the point of view of jumping numbers. Roughly stated, our main result is this: an exceptional divisor E of a minimal embedded resolution of a curve C contributes to the sequence of jumping numbers if and only if E has non-trivial intersection with at least three of the (other) components of the full transform C.

The trick of the proof is to find a jumping number that the exceptional E contributes when E satisfies the criterion, and then to apply intersection theory.

6 Multiplier ideals of monomial space curves

Multiplier ideals are notoriously hard to compute. In [18], I discover a formula for the multiplier ideals of a monomial space curve. The key to finding such a formula is a theorem of Teissier [13] and his collaborators that uses toric techniques to find embedded resolutions of one toric variety equivariantly embedded in another. I build my log resolutions from their embedded resolutions.

Future plans

Here are a few topics I am interested in:

- I am interested in computing the multiplier ideals of one (not necessarily normal) toric variety equivariantly embedded in another. Much of this work is already complete and suggests a couple of new constructions on toric varieties. I'd like to explore these.

- More generally, I'd like to find a formula for the multiplier ideals of an arbitrary binomial ideal. We have such a formula for monomial ideals (see [5]). And, a formula for one toric variety equivariantly embedded in another would be a formula for prime ideals generated by (pure) binomials. These formulas should have a common generalization.

- I am interested in the structure of multiplier ideals. While every integrally closed ideal in a two-dimensional regular local ring is a multiplier ideal, this is not true in higher dimensions. Multiplier ideals satisfy a homological condition (see [8]). It would be nice to know which ideals are multiplier ideals.

- Lately (see [2] and [11]), there has been renewed interest in my work on toric schemes. I'd like to come back to that.

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