# Mathematics for Social Scientists, II

## Calculus

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## Table of contents

1. Types of Functions and Average Rate of Change
2. The Definition of the Derivative
3. The Product, Quotient, and Chain Rules
4. The Second Derivative
5. Optimization, Exponential Functions, and Logarithmic Functions
6. Derivatives of Exponential and Logarithmic Functions
7. Partial Derivatives of Multivariate Functions
8. Integration
9. The Fundamental Theorem of Calculus

## text & references

The textbook for this section of the course is

D. Kleppner and N. Ramsey. *Quick Calculus.* Wiley, 1985.

The syllabus, CoursePack, and lecture notes for this course can be found at

http://homepages.umflint.edu/ hmthomps/icpsr/

Please read the ReadMe file.

Also see Khan Academy (https://www.khanacademy.org/math/).

# two kinds of functions

We will begin by talking about functions of one variable. The first distinction we will make between types of functions is between linear functions and nonlinear functions.

In the setting of calculus, we say a function is linear if its graph is a line. Such functions have the form

$$f(x) = mx + b$$

Here $m$ is the slope of the line and $b$ is the $y$-intercept. $y = mx + b$ is the slope-intercept form of the equation of a line.

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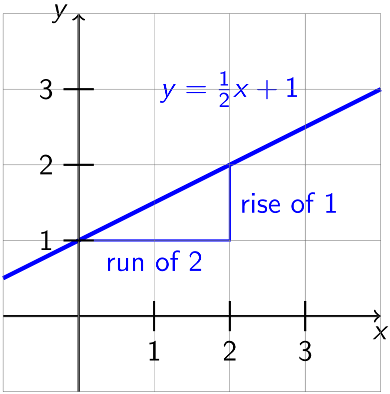
Here $m$ is the slope of the line and $b$ is the $y$-intercept. $y = mx + b$ is the slope-intercept form of the equation of a line. Recall that if $(x_1, y_1)$ and $(x_2, y_2)$ are points on a line in the Cartesian plane (with $x_1 \neq x_2$), then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

# a line

Here $m = \frac{1}{2}$ and $b = 1$.



# point-slope form

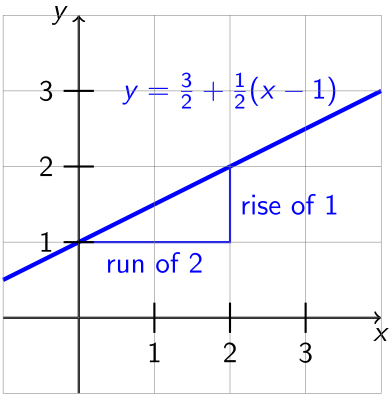
We will make use of the point-slope form of the equation for a line. If $(x_1, y_1)$ is a point on a line of slope $m$, then

$$y = y_1 + m(x - x_1)$$

is the point-slope form of the equation of that line.

# the same line

The line in the previous example has slope $m = \frac{1}{2}$ and passes through $(1, \frac{3}{2})$.



## some kinds of nonlinear functions

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- Trigonometric Functions

# average rate of change

The average rate of change of the function $f(x)$ as $x$ changes from $x_1$ to $x_2$ is the slope of the secant line through $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

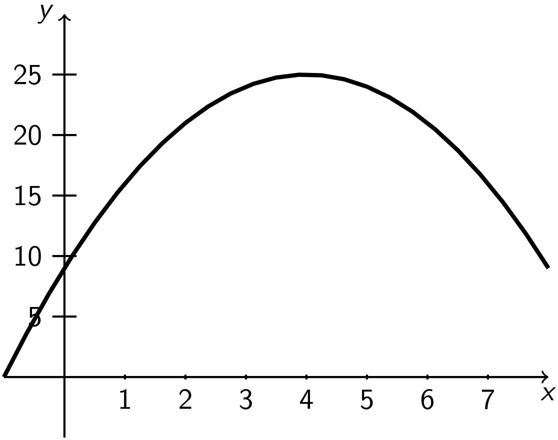
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If $y = f(x)$, we often write

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

# example

Let $y = 9 + 8x - x^2$. Find the average rate of change of $y$ as $x$ changes from 2 to 5.



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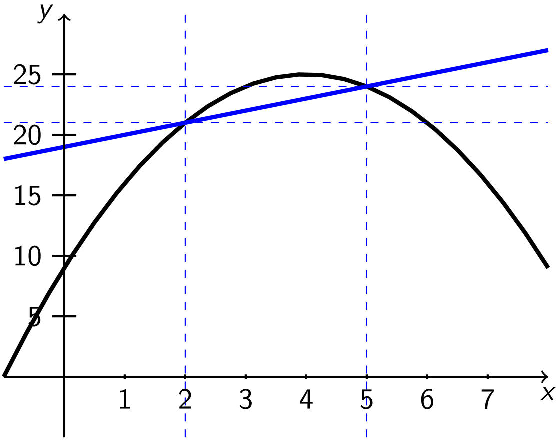
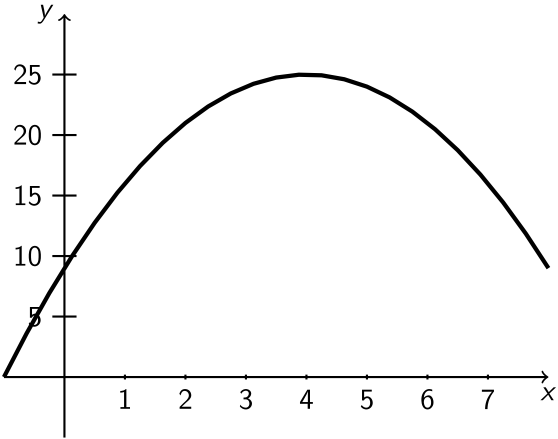


# example (cont.)

Let $y = 9 + 8x - x^2$. Find the average rate of change of $y$ as $x$ changes from 2 to 7.



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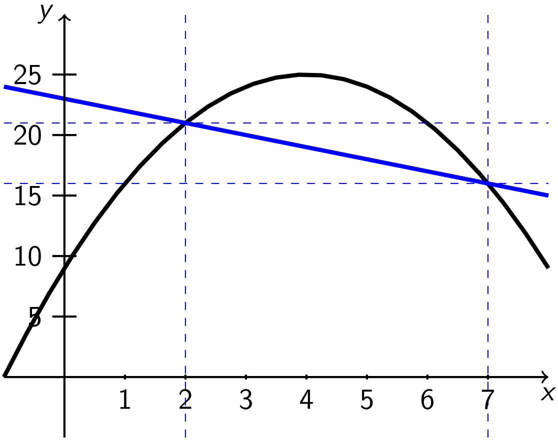
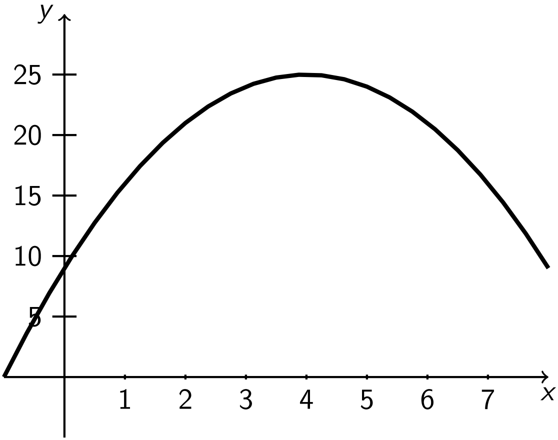


# example (cont.)

Let $y = 9 + 8x - x^2$. Find the average rate of change of $y$ as $x$ changes from 2 to 6.



# example (cont.)

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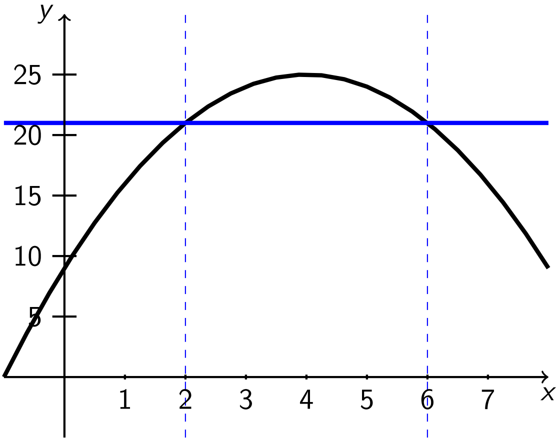

# more computations

$$\Delta x = 1 \qquad \frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{f(3)-f(2)}{3-2} = \frac{24-21}{3-2} = 3$$

$$\Delta x = 0.1 \qquad \frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{f(1.9)-f(2)}{1.9-2} = \frac{21.39-21}{1.9-2} = 3.9$$

$$\Delta x = 0.01 \qquad \frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{f(2.01)-f(2)}{2.01-2} = 3.99$$

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# even more computations

$$\Delta x = -1 \qquad \frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{f(1)-f(2)}{1-2} = \frac{16-21}{1-2} = 5$$

$$\Delta x = -0.1 \qquad \frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{f(1.9)-f(2)}{1.9-2} = \frac{20.59-21}{1.9-2} = 4.1$$

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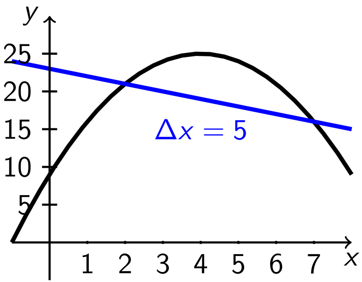
# example (cont.)

Let $y = 9 + 8x - x^2$. Let $x = 2$ and look at $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{f(2+\Delta x)-f(2)}{\Delta x}$.



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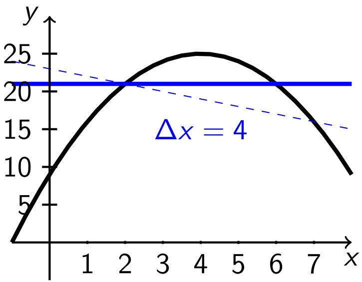

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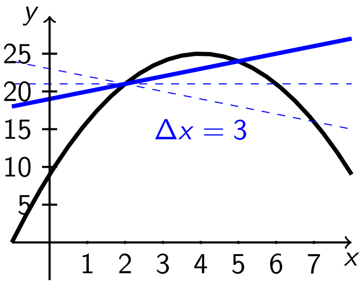

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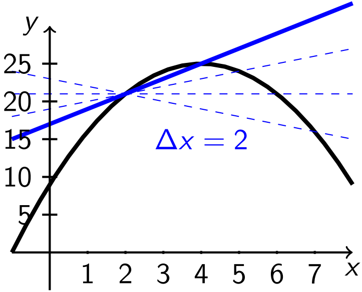

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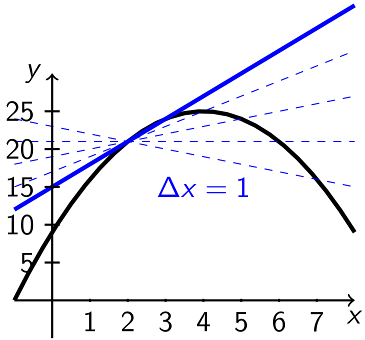

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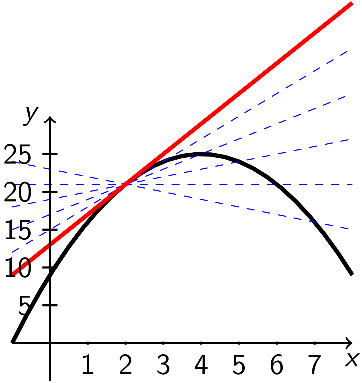


What is the slope of the tangent line at $x = 2$?

## a generalization

To get at the slope of the tangent line at $x = 2$, we study the trends in the slopes of the secant lines systematically.

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To get at the slope of the tangent line at $x = 2$, we study the trends in the slopes of the secant lines systematically.

We can simultaneously compute the slopes of all the secant lines at $x = 2$. Treat $\Delta x$ as a variable to get

$$\begin{aligned}\frac{f(2+\Delta x) - f(2)}{\Delta x} &= \frac{[9 + 8(2+\Delta x) - (2+\Delta x)^2] - [9 + 8 \cdot 2 - 2^2]}{\Delta x} \\ &= \frac{9 + 16 + 8\Delta x - 4 - 4\Delta x - (\Delta x)^2 - 9 - 16 + 4}{\Delta x} \\ &= \frac{4\Delta x - (\Delta x)^2}{\Delta x} \\ &= 4 - \Delta x\end{aligned}$$

## a generalization (cont.)

Here we see that $\frac{f(2+\Delta x)-f(2)}{\Delta x}$ tends toward 4 as $\Delta x$ tends toward zero. We write

$$\lim_{\Delta x \to 0} \frac{f(2+\Delta x)-f(2)}{\Delta x} = 4$$

and we say the <u>limit</u> of $\frac{f(2+\Delta x)-f(2)}{\Delta x}$ as $\Delta x$ approaches 0 is 4.

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and we say the limit of $\frac{f(2+\Delta x)-f(2)}{\Delta x}$ as $\Delta x$ approaches 0 is 4.

This limit is the slope of the tangent line to $y = 9 + 8x - x^2$ at the point $(2, 21)$. We also say this limit is the instantaneous rate of change of $f(x) = 9 + 8x - x^2$ at $x = 2$, or the derivative of $f(x) = 9 + 8x - x^2$ at $x = 2$.

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The tangent line to $y = 9 + 8x - x^2$ at the point $(2, 21)$ is $y = 21 + 4(x - 2) = 4x + 13$.

# the definition

More generally, we define a new function, the derivative of $y = f(x)$.

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

# the definition

More generally, we define a new function, the derivative of $y = f(x)$.

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An equation for the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is

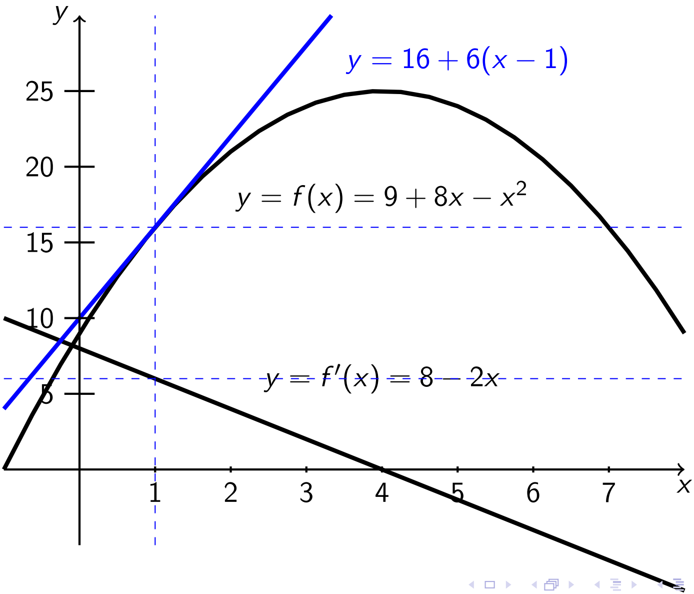
$$y = f(a) + f'(a)(x - a)$$

# our first derivative computation

For $f(x) = 9 + 8x - x^2$, we have

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[9 + 8(x+\Delta x) - (x+\Delta x)^2] - [9 + 8x - x^2]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{9 + 8x + 8\Delta x - x^2 - 2x\Delta x - (\Delta x)^2 - 9 - 8x + x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{8\Delta x - 2x\Delta x - (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (8 - 2x - \Delta x) \\
&= 8 - 2x
\end{aligned}
$$

# example (cont.)



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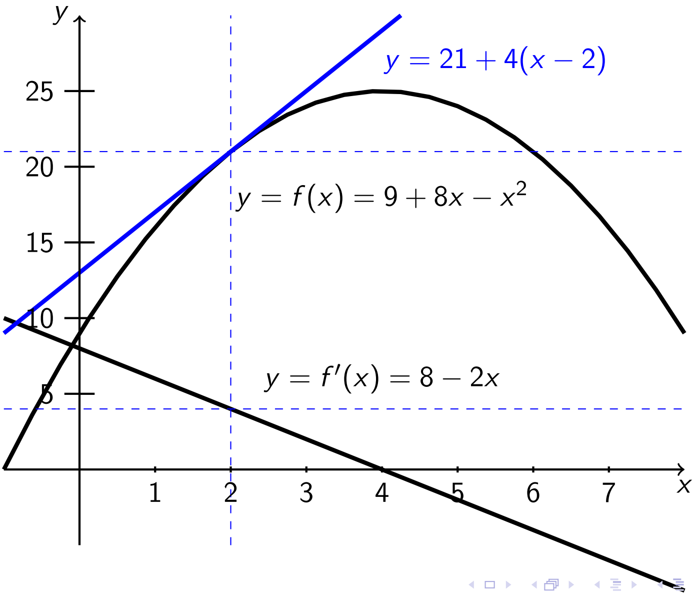

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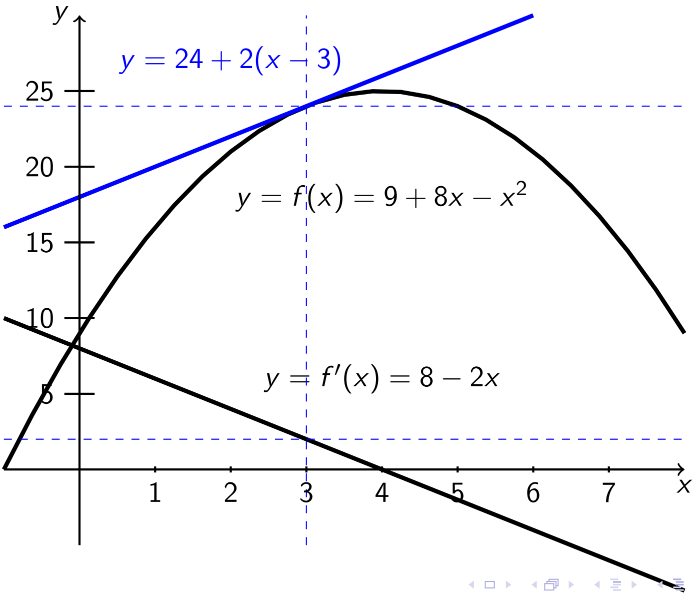

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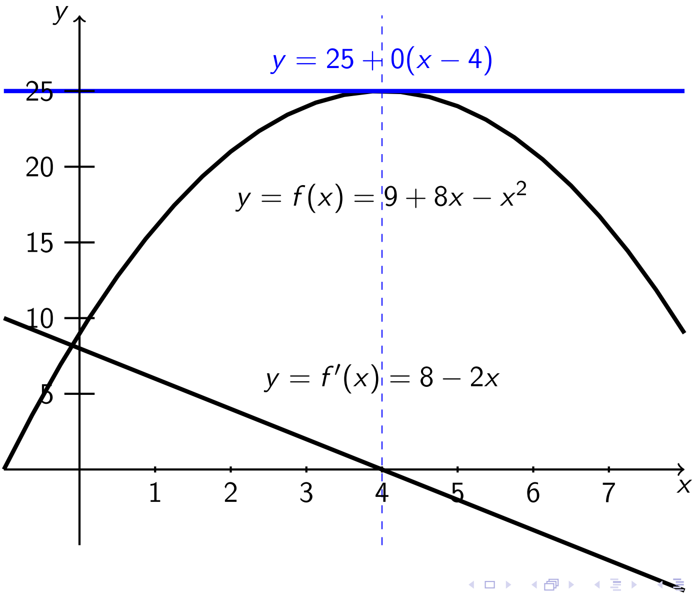

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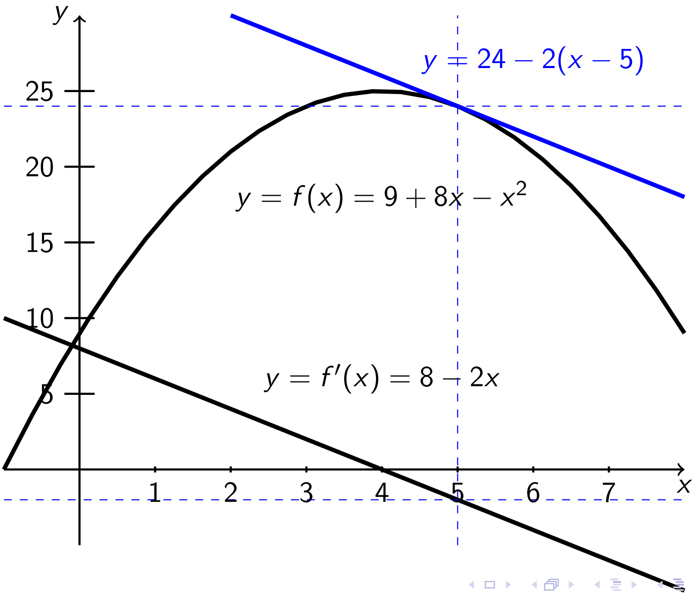

# the first few rules

$$y = f(x) = c \qquad \frac{dy}{dx} = f'(x) = 0$$

$$y = f(x) = x^n \qquad \frac{dy}{dx} = f'(x) = nx^{n-1}$$

$$y = cf(x) \qquad \frac{dy}{dx} = cf'(x)$$

$$y = f(x) + g(x) \qquad \frac{dy}{dx} = f'(x) + g'(x)$$

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Use the rules above to compute the derivative of the following function.

$y = f(x) = mx + b$

# the first few rules

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$$

Use the rules above to compute the derivative of the following function.

$y = f(x) = 9 + 8x - x^2$

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$$y = f(x) + g(x) \qquad \frac{dy}{dx} = f'(x) + g'(x)$$

Use the rules above to compute the derivative of the following function.

$$y = f(x) = -0.24x^3 + 6.91x^2 + 1.73x + 9.58$$

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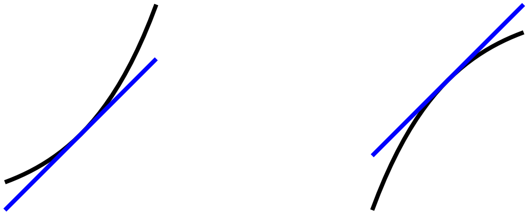
$$y = f(x) + g(x) \qquad \frac{dy}{dx} = f'(x) + g'(x)$$

Use the rules above to compute the derivative of the following function.

$y = f(x) = \frac{7}{\sqrt[4]{x^3}}$

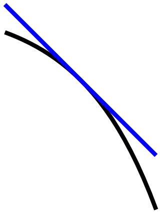
# another definition

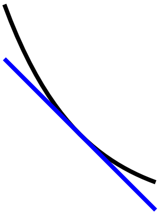
If $f'(a) > 0$, we say $f$ is increasing at $x = a$.



# another definition

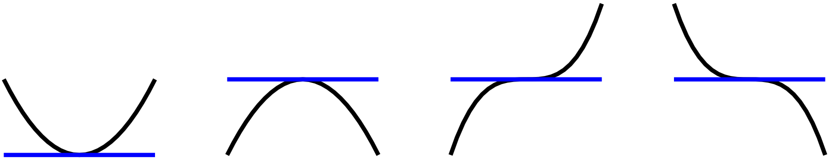
If $f'(a) < 0$, we say $f$ is decreasing at $x = a$.





# another definition

If $f'(a) = 0$, we say $f$ is stationary at $x = a$.



# practice

Find the intervals of increase and decrease for the following function.

$f(x) = 9 + 8x - x^2$

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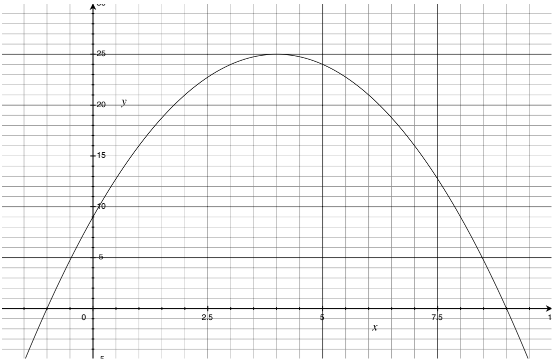

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Find the intervals of increase and decrease for the following function.

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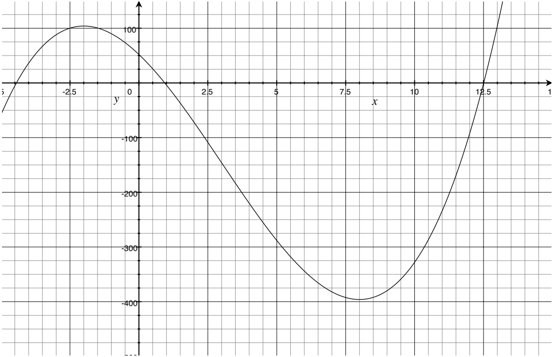

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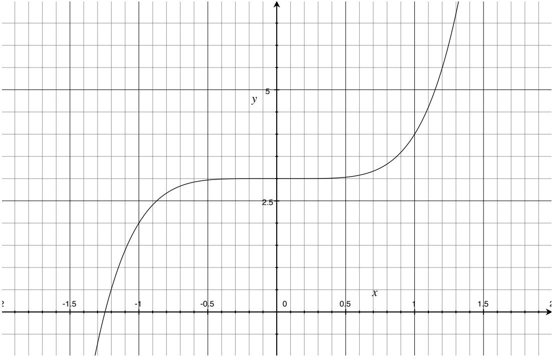

# product rule

If $y = f(x) = u(x)v(x)$, then

$$y' = f'(x) = \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = u'v + uv'$$

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Try this for $y = x^2 \cdot x^3$.

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Try this for $y = (x^4 + 7x^2 - 5)(x^5 + 3x^3 + x - 2)$.

## quotient rule

If $y = f(x) = \frac{u(x)}{v(x)}$, then

$$y' = f'(x) = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{u'v - uv'}{v^2}$$

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Try this for $y = \frac{x^3}{(1-x)^3+x^3}$.

# chain rule

If $y = g(x) = f(u(x))$, then

$$y' = g'(x) = \frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = f'(u(x)) \cdot u'(x)$$

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Try this for $y = (3x+1)^2$.

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Try this for $y = \sqrt[3]{x^2 + 1}$.

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Try this for $y = \frac{1}{(1-x^2)^4}$.

## definition

We call the derivative of the derivative the second derivative. If $y = f(x)$,

$$y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

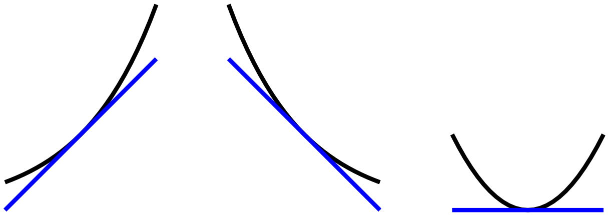
It is the instantaneous rate of change of the slope of the tangent line to the curve $y = f(x)$.

## physics

If $x = f(t)$ is the position of some object at time $t$, what is physical the meaning of $f'(t)$ and $f''(t)$?

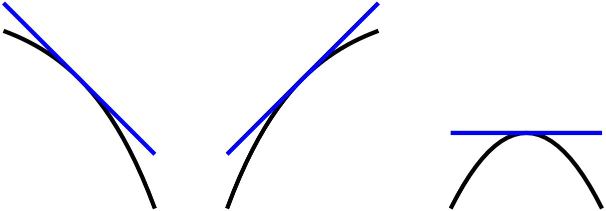
# concavity

If $f''(a) > 0$, we say $f$ is concave up at $x = a$.



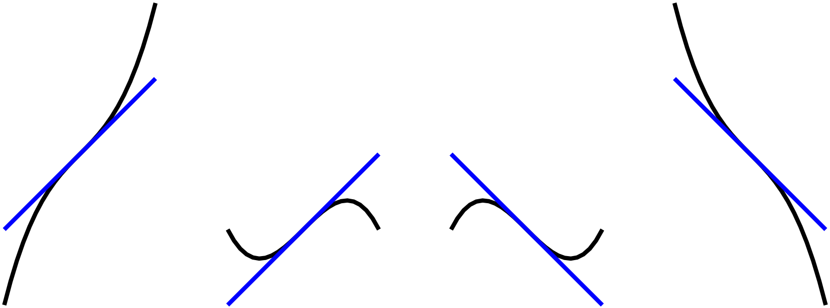
# concavity

If $f''(a) < 0$, we say $f$ is concave down at $x = a$.



# concavity

We say $f(x)$ has an inflection point at $x = a$ if the concavity changes at $x = a$.



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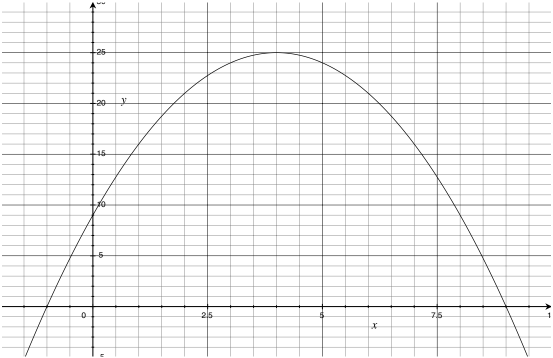

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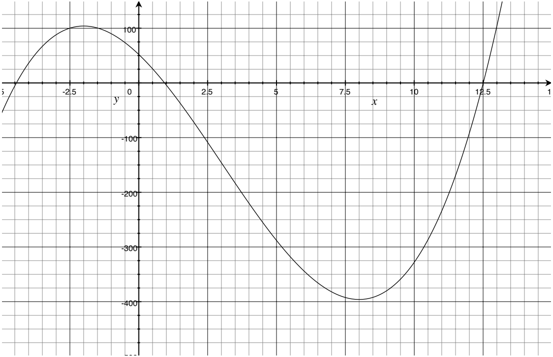

## practice

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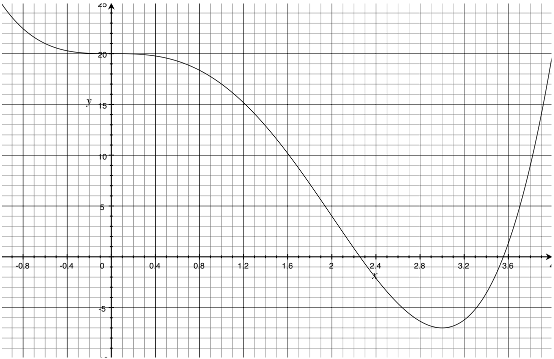

# from the radio show Marketplace

From Friday, June 5, 2009

Tess Vigeland: I guess maybe they're really looking at these short-term indications, you know, unemployment did not drop as much as previous months and the housing market seems to be picking up, but...

Felix Salmon: Yeah, this is one of my favorite pieces of optimism, where people look at the second derivative of the unemployment numbers. They don't look at how enormously high it is. They don't say, "It's 9.4 percent! That's horrible!" Instead they look at the difference between the number of people who lost their jobs in May and the number of people who lost their jobs in April. And they say "Oooooh, things are turning around." This is just never convincing to me.

# from the radio show Marketplace

From Friday, January 31, 2014

This is Felix Salmon responding to Kai Ryssdals pessimism about a slow in the increase of the GDP:

"I'm going to be optimistic on the first derivative, and you can be pessimistic on the second derivative, and I'm going to be the person going up, and you're going to be going up as well, just not realizing it."

# optimization

What can be said about a function $f(x)$ at a point $x = a$ if $f$ is stationary at $x = a$, $f'(x) > 0$ to the left of $x = a$, and $f'(x) < 0$ to the right of $x = a$?

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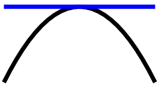

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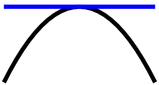

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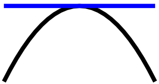

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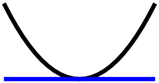

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- In the First Derivative Test, we check $f'(x)$ on each side of $a$.
- In the Second Derivative Test, we check $f''(a)$.

Explain the details to another participant.

## practice

Find the local extrema of the following function.

$f(x) = x^3 - 6x^2 + 9x$

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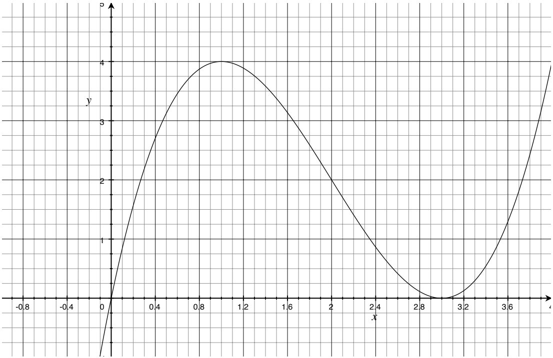

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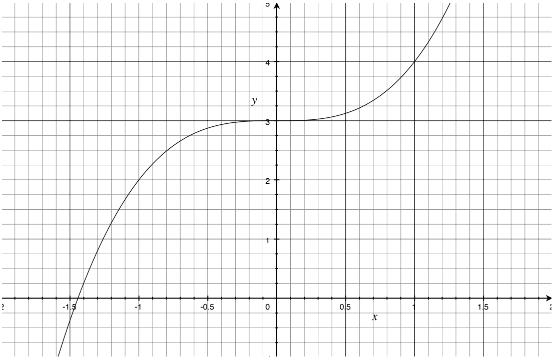

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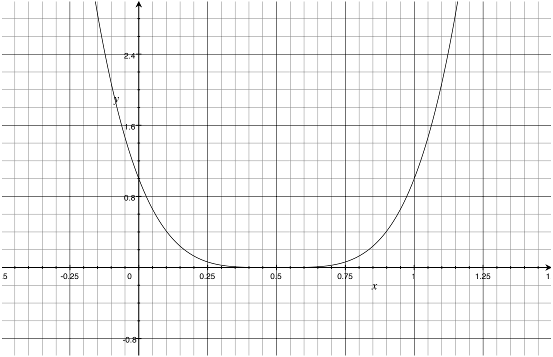
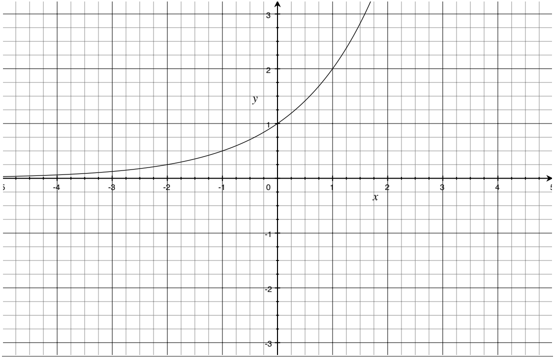


# exponential functions

Fix a number $b > 0$ with $b \neq 1$. An exponential function is a function of the form $f(x) = b^x$.

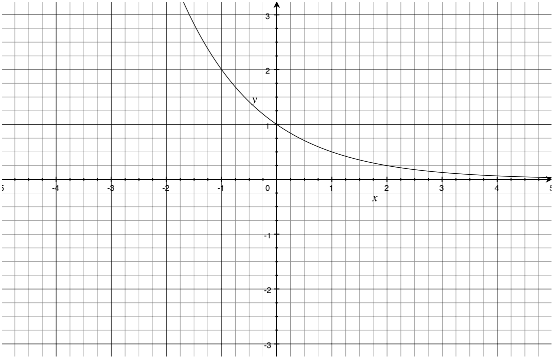
$y = 2^x$



# exponential functions

Fix a number $b > 0$ with $b \neq 1$. An exponential function is a function of the form $f(x) = b^x$.

$y = \left(\frac{1}{2}\right)^x$



# properties

Recall:

- $b^{x_1+x_2} = b^{x_1} \cdot b^{x_2}$

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- $b^{px} = (b^x)^p$

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- $b^{px} = (b^x)^p$

In particular, $b^{-x} = \left(\frac{1}{b}\right)^x$.

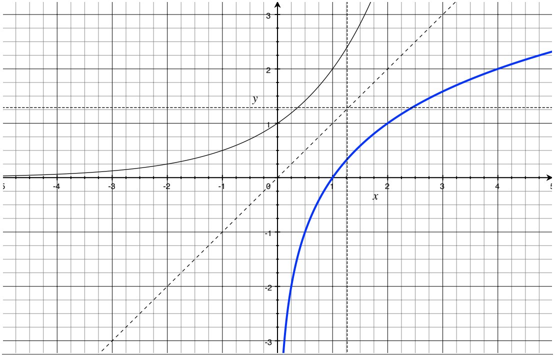
# model question

Name some things that are modeled by exponential functions.

# logarithmic functions

Fix a number $b > 0$ with $b \neq 1$. The logarithmic function with base $b$ is the function $f(x) = \log_b x$ such that

$$y = \log_b x \text{ if and only if } b^y = x$$

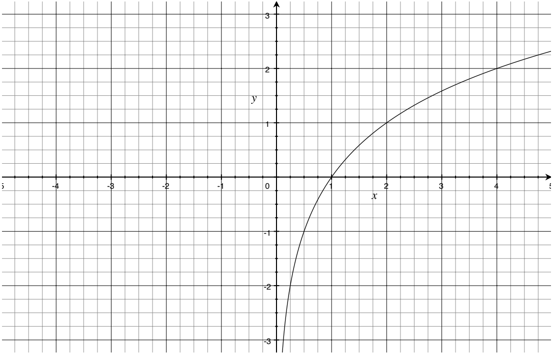


# logarithmic functions

We write $\log x = \log_{10} x$ for the common logarithm and $\ln x = \log_e x$ for the natural logarithm.

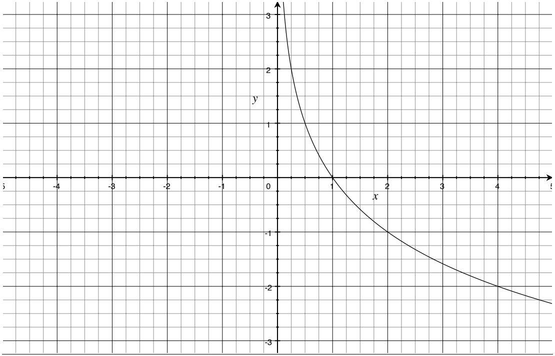
# logarithmic functions

$y = \log_2 x$



# logarithmic functions

$$y = \log_{\frac{1}{2}} x$$



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- $\log_b x_1 - \log_b x_2 = \log_b \left(\frac{x_1}{x_2}\right)$
- $p \log_b x = \log_b x^p$

# model question

Name some things that are modeled by logarithmic functions.

# derivatives

If $y = e^{u(x)}$, then

$$\frac{dy}{dx} = e^{u(x)} \cdot \frac{du}{dx}$$

# practice

Find the intervals of increase and decrease, and the intervals of concavity for the following function.

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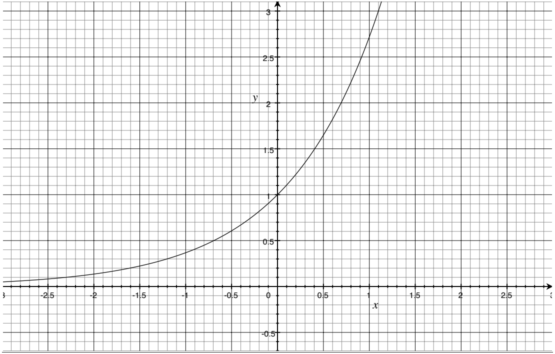

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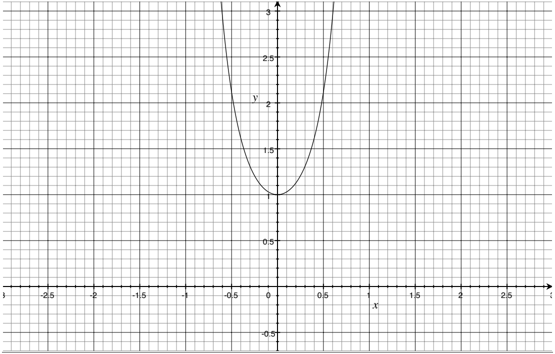

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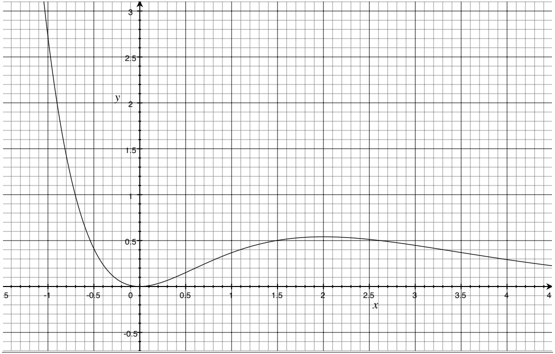

# practice

Find the intervals of increase and decrease, and the intervals of concavity for the following function.

$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where $\mu$ and $\sigma$ are constants

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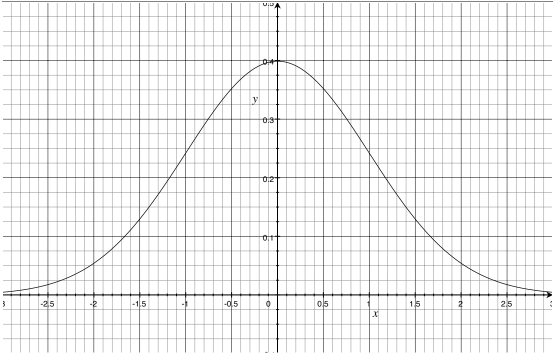

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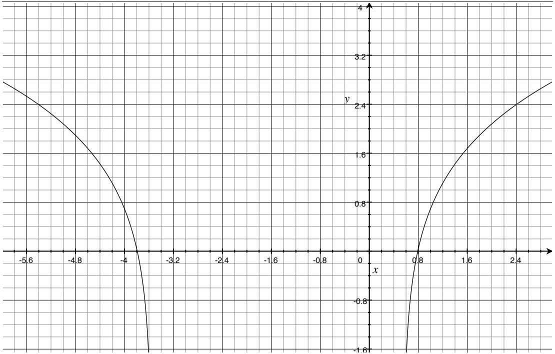

# quadratic approximation

We have seen linear approximation. A function is approximated, near $x = a$, by the linear function with the same value and derivative at $a$.

$$f(x) \approx f(a) + f'(a)(x - a)$$

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$$f(x) \approx f(a) + f'(a)(x-a)$$

We can also approximate $f$ by the quadratic function that has the same value and first two derivatives at $x = a$.

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Such approximations are used to estimate the location of the closest extremum based on local data. In particular, we estimate the extremum is the extremum of the quadratic.

# quadratic approximation

We have seen linear approximation. A function is approximated, near $x = a$, by the linear function with the same value and derivative at $a$.

$$f(x) \approx f(a) + f'(a)(x - a)$$

We would get the same estimate if we worked with the linear approximation of the derivative.

$$f'(x) \approx f'(a) + f''(a)(x - a)$$

With this approach, we find where this linear function is zero.

# quadratic approximation

Instead of using quadratic approximation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

directly, we could get the same estimate if we worked with the linear approximation of the derivative.

$$f'(x) \approx f'(a) + f''(a)(x-a)$$

With this approach, we find where this linear function is zero.

# quadratic approximation example

Find the linear approximation to

$$f(x) = x^3 - x + 3$$

at $x = -1$.

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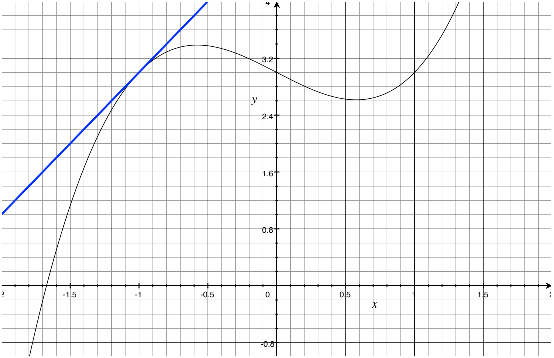

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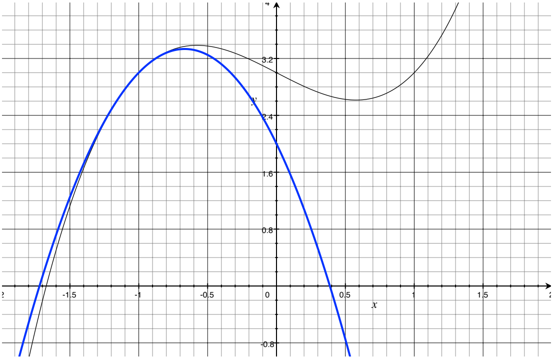

# quadratic approximation example

We estimate that the maximum occurs at $x = -\frac{2}{3}$. Note that the local maximum actually occurs when $f'(x) = 0$. That is, when $x = -\sqrt{\frac{1}{3}} \approx -0.577$. If we wanted to improve this estimate of the location of the maximum, we could do another quadratic approximation at $x = -\frac{2}{3}$. That is, we could iterate the process at the previous estimate.

We would get the same estimate if we worked with the linear approximation of the derivative

$$y = f''(-1)(x+1) + f'(-1) = -6x - 4$$

# partial derivatives

Let $y = f(x_1, x_2, \ldots, x_n)$ be a function of more than one variable. The partial derivative of $f$ with respect to $x_i$ is

$$\begin{aligned}\frac{\partial y}{\partial x_i} &= f_{x_i} \\ &= \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + \Delta x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}\end{aligned}$$

Find the partial derivatives of the following function.

$Y = 6 - 2X_1 - 3X_2$

# partial derivatives

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Is there anything worth noticing?

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Check that $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + C$ is an antiderivative of $\frac{dy}{dx} = x^2 + x - 6$ for any choice of the constant $C$.

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The notation for this general antiderivative or indefinite integral is

$$\int \left(x^2 + x - 6\right)\, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + C$$

# indefinite integrals

Find the indefinite integral.

$$\int \left(x^3 - 2x + 5\right)\, dx$$

## indefinite integrals

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$$\int \left( x^{\frac{1}{3}} + x^{-\frac{1}{2}} \right)\, dx$$

# indefinite integrals

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$$\int e^x \, dx$$

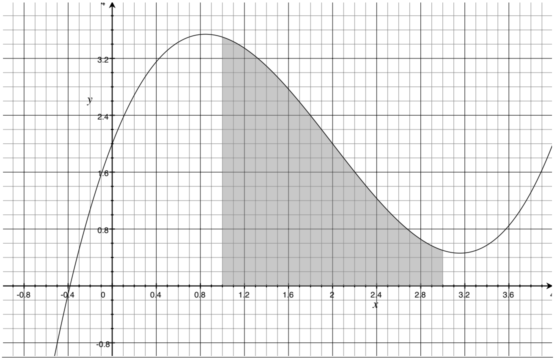
Find the indefinite integral.

$$\int \frac{1}{x}\, dx$$

Do the exponential growth example.

# area problem

Problem: Suppose, for the sake of simplicity, that $f(x) > 0$. Find the area trapped below the curve $y = f(x)$, above the $x$-axis, and between the lines $x = a$ and $x = b$.



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$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$A = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

when this limit exists independent of the choices of partitions and the $x_i^*$.

We call this limit the definite integral and write

$$A = \int_a^b f(x)\, dx$$

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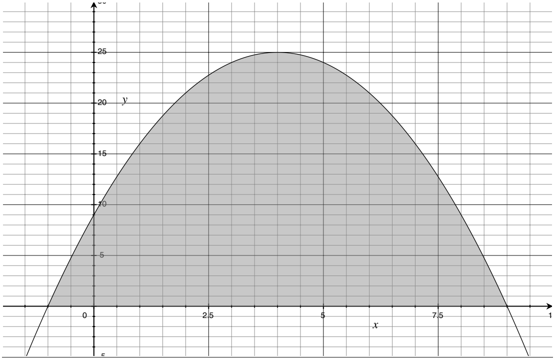
Notice that, these areas are signed. In particular, area below the $x$-axis is negative.

# definite integrals

Find the definite integral.

$$\int_{-1}^{9} \left(9 + 8x - x^2\right)\, dx$$



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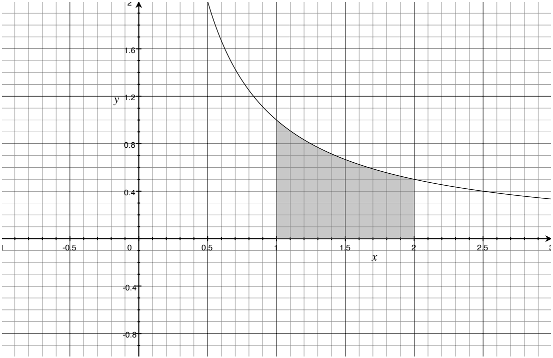
$$\int_1^2 \frac{1}{x}\, dx$$



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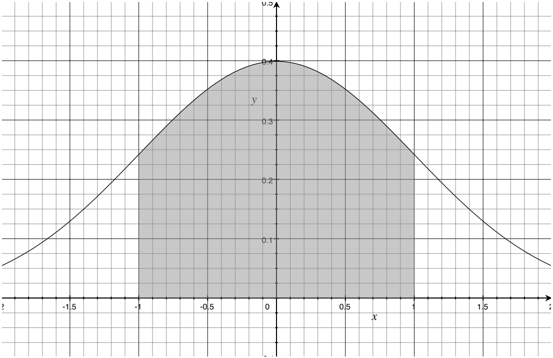
$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \approx 0.6827$$



# definite integrals

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \approx 0.6827$$

This value can be estimated by sums despite the fact that the antiderivative of $e^{-\frac{1}{2}z^2}$ is not an elementary function (cannot be constructed using combinations sums, differences, products, quotients, and compositions of exponentials, logarithms, and power functions).