Quadtratic Approximation Example

We will consider the following function:

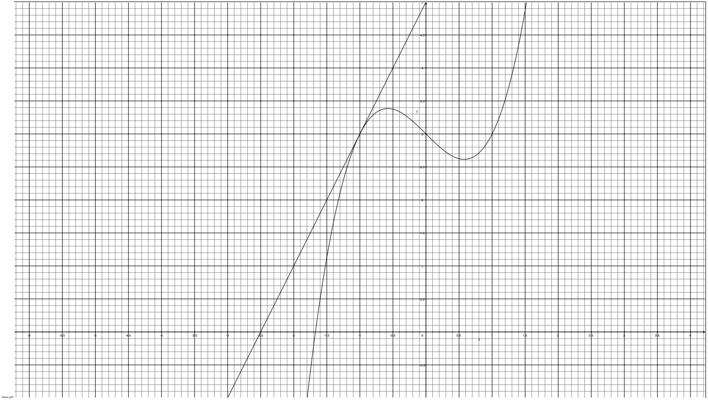
$$y = f(x) = x(x+1)(x-1) + 3 = x^3 - x + 3$$

when $x = -1$. Note that

$$f'(x) = 3x^2 - 1.$$

So, $f'(-1) = 2$ and the equation of the tangent line at $x = -1$ is

$$y = f'(-1)(x+1) + f(-1) = 2x + 5$$

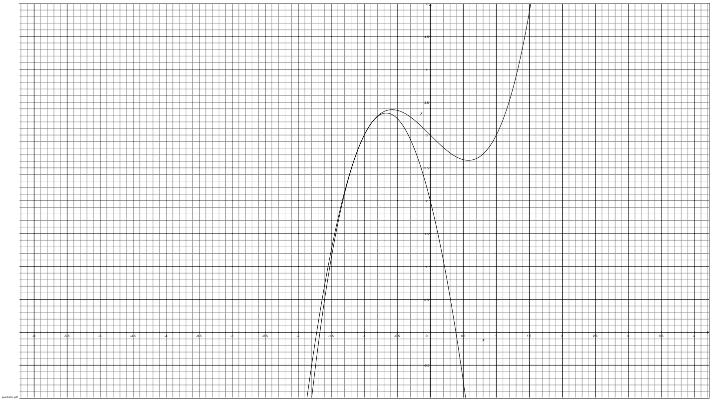


$$y = x^3 - x + 3 \text{ and } y = 2x + 5$$

Since $f''(x) = 6x$, $f''(-1) = -6$ and the quadratic approximation of $f(x)$ at $x = -1$ is

$$y = \frac{f''(-1)}{2}(x+1)^2 + f'(-1)(x+1) + f(-1)$$
$$= -3x^2 - 4x + 2 = -3\left(x + \frac{2}{3}\right)^2 + \frac{10}{3}$$

So, if we were to estimate where the local maximum of $f(x)$ from the first two derivatives at $x = -1$, we would guess it is at the maximum of the quadratic approximation. This estimate yields the x-value $-\frac{2}{3} \approx -0.667$.



$$y = x^3 - x + 3 \text{ and } y = -3x^2 - 4x + 2$$

Note that the local maximum actually occurs when $f'(x) = 0$. That is, when $x = -\sqrt{\frac{1}{3}} \approx -0.577$. If we wanted to improve this estimate of the location of the maximum, we could do another quadratic approximation at $x = -\frac{2}{3}$. That is, we could iterate the process at the previous estimate.

We would get the same estimate if we worked with the linear approximation of the derivative

$$y = f''(-1)(x+1) + f'(-1) = -6x - 4$$