MATRICES

Word or notation: around 1850 (Sylvester; English)

MATRIX : from Latin for womb
Beautiful word (generator or carrier)

Caution: someone offended when I said we can "cram information into a matrix"

- Rectangular array of numbers
- See example sheet
- notation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Some use ( ) instead of [ ]

NOT: | |

- order $m \times n$ (↑ rows, ↑ columns)

ex 1 is 5×3

ex 2 is 8×8

- elements a_{ij}
- $A = [a_{ij}]$ Boldface type in text and ~~[illegible]~~ problems. I'll not take the time on board
- Equality $\underset{m\times n}{A} = \underset{m\times n}{B}$ if $a_{ij} = b_{ij}$ for each i and j.
- Some special matrices

1. Zero : O $\quad a_{ij} = 0$
2. Square : $m = n$ (ex. 2)
3. diagonal : square with $a_{ij} = 0$ for $i \neq j$ $\quad \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix}$
4. Identity : diagonal with $d_i = 1$ $\quad I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$

Problems 1, 2, 3

EXAMPLES OF MATRICES

Back to these later!

1. Data Storage

$$
\begin{array}{cc}
\text{Employee Number} & \begin{array}{ccc}\text{Days Absent} & \text{Attitude} & \text{Experience}\end{array} \\
\begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & 2 \\ 6 & 7 & 4 \\ 9 & 10 & 8 \end{bmatrix}
\end{array}
$$

Psychologist study

(attitude : 1 = good)

2. Dominance [S.S.Ulmer, "Leadership in the Michigan Supreme Court," *Judicial Decision Making*, 1963]

$$
D = \begin{array}{lc}
 & \begin{array}{cccccccc} V & Ka & D & C & S & E & Ke & B \end{array} \\
\begin{array}{l}\text{Volker}\\ \text{Kavanaugh}\\ \text{Detmers}\\ \text{Carr}\\ \text{Smith}\\ \text{Edwards}\\ \text{Kelly}\\ \text{Black}\end{array} &
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\qquad
d_{ij} = \begin{cases} 1 & \text{if } i \text{ dominates } j \\ 0 & \text{if } i \text{ does not dominate } j \end{cases}
$$

Political Scientist

Carr supports more of Kavanaugh's decisions than Ka support C
i.e. Ka "dominates" C

3. Markov Process [Glass & Hall, "Social Mobility in Great Britain: A Study of Intergeneration Changes in Status," *Social Mobility in Great Britain*, 1954]

$$
\begin{array}{lc}
 & \begin{array}{ccc} U & M & L \end{array} \\
\begin{array}{l}\text{Upper}\\ \text{Middle}\\ \text{Lower}\end{array} &
\begin{bmatrix} .448 & .484 & .068 \\ .054 & .699 & .247 \\ .011 & .503 & .486 \end{bmatrix}
\end{array} = \mathbf{P}
$$

where p_{ij} is the probability that a son of a person working in an occupation of class i gets a job in an occupation of class j

Demographer

U = upper (e.g. "govt. official")
M = middle (e.g. teacher)
L = lower (e.g. laborer)

4. System of Linear Equations

Data:

x	y	z
0	575	50
50	857	406
90	397	994
100	348	1016

Linear Model: $z = a+bx+cy$

Simultaneous Equations

$$
\begin{aligned}
a + + 575c &= 50 \\
a + 50b + 857c &= 406 \\
a + 90b + 397c &= 994 \\
a + 100b + 348c &= 1016
\end{aligned}
$$

Sociologist

Matrix Representation

$$
\left[\begin{array}{ccc:c} 1 & 0 & 575 & 50 \\ 1 & 50 & 857 & 406 \\ 1 & 90 & 397 & 994 \\ 1 & 100 & 348 & 1016 \end{array}\right]
$$

x = year (x = 0 is 1900)
y = Pop (x1000) of St. Louis City
z = Pop. (x1000) of St Louis County (incl city)

x = 103 (2003) ⟹ y = 332

Note: Since 1950, St Louis city has the greatest percentage loss of any major city in the world in modern times.

Some Operations on Matrices

97

① addition

- A, B both $m \times n$
- $A+B = [a_{ij} + b_{ij}]$
- $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$ $\quad A+B = \begin{bmatrix} 3 & -1 & 4 \\ 5 & -1 & 0 \end{bmatrix}$

② Subtraction

- A, B both $m \times n$
- $A - B = [a_{ij} - b_{ij}]$
- A, B as above $\quad A - B = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 1 & -4 \end{bmatrix}$

Problems 4, 5

③ Scalar Multiplication

- $kA = [k a_{ij}]$
- $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad 2A + B = \begin{bmatrix} 3 & 0 \\ 6 & -1 \\ 6 & -2 \end{bmatrix}$

Problem 6, 7

DAY 1

Review Day 2 $\quad A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$ $\quad C = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

$$2A - B + C = \begin{bmatrix} 0 & -4 \\ 3 & 2 \\ 8 & 1 \end{bmatrix}$$

"QUIZ" $\quad \underbrace{3\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{\text{square}} + \begin{bmatrix} -1 & p \\ -3 & 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 2 & 0 \\ q & 1 \end{bmatrix}}_{\text{diagonal}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{zero}}$ $\quad \therefore p = -6, \; q = 0$

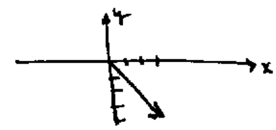
Vectors

$1 \times n$ matrix is a row vector ; $n \times 1$ matrix is a column vector.

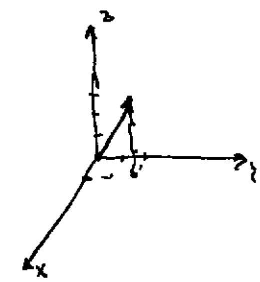
$\therefore$ Equality, addition, subtraction, scalar multiplication of vectors is same as for matrices.

note: if $n=2$ or $n=3$ we have "geometric models".

e.g. $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ or $[3, -4]$



$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$



use straw for $n=3$

Inner (dot) Product $a \cdot b$

a' is a row vector $\quad a' = [a_1 a_2 \cdots a_n]$ $\quad$ (ie transpose of $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$)

b is a column vector $\quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

same n.

$$a' \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Note: This is a number (scalar)

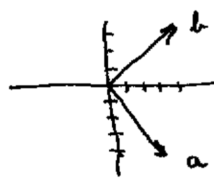
EXAMPLES

① (i) $a' = [1 \ 2 \ 3] \quad b = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \quad a' \cdot b = 1(-1) + 2(3) + 3(5) = 20$

(ii) $a' = [1 \ 2 \ 3] \quad a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad a' \cdot a = 1^2 + 2^2 + 3^2 = 14$ (sum of squares).

② $a = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \therefore a' \cdot b = [3 \ -4] \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 0$

Note geometrically:



The vectors are perpendicular (orthogonal).
also the length of each vector is $\sqrt{3^2+4^2} = 5$

Def: ① a and b are orthogonal if $a' \cdot b = 0$ [not just in 2 or 3 dim]

② The length of a is $\sqrt{a' \cdot a} = |a|$

③ a vector is a unit vector if $|a| = 1$ $\quad$ N.B. More general than def. on p.50 in Namb.

Note: if $|a| = 5$ then a unit vector would be $\frac{1}{5}$ of a, ie $\frac{1}{5} a$ or $\frac{1}{|a|} a$.

e.g. $a' = [1 \ 2 \ -3] \quad |a| = \sqrt{14} \quad \therefore \frac{1}{\sqrt{14}} a' = \left[\frac{1}{\sqrt{14}} \ \frac{2}{\sqrt{14}} \ \frac{-3}{\sqrt{14}}\right]$ is a unit vector in same direction as a

Matrix Multiplication

$$\underset{m\times p}{A}\ \underset{p\times n}{B} = \underset{m\times n}{C}$$

"conformable for mult"

c_{ij} is the inner product of the i^{th} row of A and the j^{th} column of B ("shirt" or "diving board")

EXAMPLES:

① $\underset{\substack{2\times 3\\ A}}{\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}} \underset{\substack{3\times 4\\ B}}{\begin{bmatrix}1&0&2&0\\0&1&2&1\\1&0&1&2\end{bmatrix}} = \underset{\substack{2\times 4\\ C}}{\begin{bmatrix}4&2&9&8\\10&5&24&17\end{bmatrix}}$ e.g. $c_{23} = [4\ 5\ 6]\begin{bmatrix}2\\2\\1\end{bmatrix} = 4(2)+5(2)+6(1)$

note: BA is not defined!

② p23 in text (note error - remove the .144)

$A = \begin{bmatrix}1&2\\3&4\end{bmatrix}$ $B = \begin{bmatrix}5&6\\7&8\end{bmatrix}$ $\therefore AB = \begin{bmatrix}19&22\\43&50\end{bmatrix}$

$BA = \begin{bmatrix}5&6\\7&8\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix} = \begin{bmatrix}23&34\\31&46\end{bmatrix}$

So even if AB and BA are both defined they may not be equal.

③ $A = \begin{bmatrix}1&2\\3&7\end{bmatrix}$ $B = \begin{bmatrix}7&-2\\-3&1\end{bmatrix}$ $AB = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$ (identity matrix)

$BA = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$

So here $AB = BA = I$

(is there something like an inverse ("division")?)

④ Why is multiplication defined as it is?

Recall example 3 (markov process - intergenerational changes)

$$P = \begin{array}{c|ccc} & U & M & L\\ \hline U & .448 & .484 & .068\\ M & .054 & .699 & .247\\ L & .011 & .503 & .486\end{array}$$

p_{ij} = probability a son of a man working in class i will get a job in class j.

Let $X = \begin{array}{ccc} U & M & L\\ {[76} & 634 & {290]}\end{array}$ be the distribution of 1000 men in 1949

How many in each class in next generation?

e.g. U comes from 3 sources: U, M, L.

$.448(76) + .054(634) + .011(290) = 71.474$. Similarly for M and L. XP does it

$XP = [71.474\ \ 625.82\ \ 302.706] \approx [71, 626, 303]$

DAY 2

$XP^2 = (XP)P \approx [69\ \ 624\ \ 307]$

More Examples

⑤ System of Linear Equations (see example sheet #4).

$$a + .575c = 50$$
$$a + 50b + 857c = 406$$
$$a + 90b + 397c = 994$$
$$a + 100b + 48c = 1016$$

$$\begin{bmatrix} 1 & 0 & 575 \\ 1 & 50 & 857 \\ 1 & 90 & 994 \\ 1 & 100 & 348 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 50 \\ 406 \\ 994 \\ 1016 \end{bmatrix}$$

$$Ax = b$$

$\vdots$

$$ay = b$$

$\vdots$

$$3x = 15$$

④ - Example sheet #2 · Path analysis

"Dominance" [primate colony, people at a long meeting, prisoners, research vessel, etc]

Path Analysis

Example [S.S. Ulmer, "Leadership in the Michigan Supreme Court," *Judicial Decision-Making*, G. Schubert (ed.), 1963, 13-28.]

$$
\begin{array}{c}
\\ V \\ Ka \\ D \\ C \\ S \\ E \\ Ke \\ B
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} V & Ka & D & C & S & E & Ke & B \end{array} \\
\left[\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 6 \\ 6 \\ 5 \\ 5 \\ \\ \\ \\ \\ \end{array}
= \mathbf{D} \text{ (dominance)}
$$

$$
d_{ij} = \begin{cases} 1 & \text{if } i \text{ dominates } j \\ 0 & \text{otherwise} \end{cases}
$$

Justices in the Michigan Supreme Court, 1958-1960:

V = Voelker, Ka = Kavanaugh, D = Dethmers, C = Carr, S = Smith, E = Edwards, Ke = Kelly, B = Black.

2nd row (Ka), 6th column (E)

Then

$$
\mathbf{D}^2 =
\begin{array}{c} \\ Ka \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccccc} & & & & & & D & \end{array} \\
\left[\begin{array}{cccccccc}
0 & 0 & 2 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 0 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 1 & 1 & 2 & 3 & 4 \\
1 & 0 & 0 & 0 & 1 & 2 & 3 & \boxed{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 17 \\ 14 \\ 12 \\ 11 \\ \\ \\ \\ \\ \end{array}
$$

Ka → D → E

Ka → C → E

Ka → S → E

∴ Ka "dominates" E *indirectly* in 3 ways

$$
\text{so} \quad \mathbf{D} + 1/2\,\mathbf{D}^2 =
\left[\begin{array}{cccccccc}
0 & 1 & 1 & 1.5 & 2 & 2.5 & 3 & 3.5 \\
.5 & 0 & 1.5 & 1 & 2 & 2.5 & 3 & 3.5 \\
1 & .5 & 0 & .5 & 1.5 & 2 & 2.5 & 3 \\
.5 & 0 & 1 & 0 & 1.5 & 2 & 2.5 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1.5 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{c} 14.5 \\ 14.0 \\ \\ \\ \\ \\ \\ \\ \end{array}
$$

Direct + ½ indirect.

Could also look at

$$D + \tfrac{1}{2}D^2 + \tfrac{1}{3}D^3 \quad \text{etc.}$$

Power of i.

See *Matrices and Society*, Bradley & Meek, pp 86–91

Transpose of a Matrix

A' (or A^T or A^t) is the matrix obtained from A by interchanging the rows and columns of A, i.e. if $A = [a_{ij}]$ then $A' = [a_{ji}]$

e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\quad A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Problem # 13, 14, 15

note: $A'A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$

sum of squares

sum of cross products

$\cancel{AA' = \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \\ 17 & 39 & 61 \end{bmatrix}}$

Recall: $a = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix} \Rightarrow a' = [2 \;\; -4 \;\; 3 \;\; 1]$

SOME MATRIX ALGEBRA PROPERTIES (THEOREMS)

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, if defined — commutative
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$, if defined — associative
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$, where $\mathbf{0}$ is a zero matrix
4. $\mathbf{A} - \mathbf{A} = \mathbf{0}$, where $\mathbf{0}$ is a zero matrix
5. If $\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$ then $\mathbf{B} = \mathbf{C}$

   Proof: $-A+(A+B) = -A+(A+C)$
   $(-A+A)+B = (-A+A)+C$
   $0+B = 0+C$
   $B = C$
6. $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$
7. $k\,\mathbf{0} = \mathbf{0}$, where $\mathbf{0}$ is a zero matrix and k is any real number
8. $0\,\mathbf{A} = \mathbf{0}$, where 0 is the scalar zero and $\mathbf{0}$ is a zero matrix
9. $-(-\mathbf{A}) = \mathbf{A}$
10. $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$

    note: A, B are $m \times n$
    LHS has mn additions and mn multiplications
    RHS has $mn + mn$ mult and mn additions.
    ∴ LHS has $2mn$ operations and RHS has $3mn$.
    ∴ LHS shorter
    Note: mult more costly
    LHS has mn
    RHS has $2mn$
11. $k_1(k_2\mathbf{A}) = (k_1k_2)\mathbf{A}$
12. $(k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$
13. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, if defined — assoc.
14. $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$ — move scalar
15. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$; $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$, if defined — Distributive
16. $(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2$
17. $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$, where $\mathbf{I}$ is the appropriate identity matrix
18. $\mathbf{0A} = \mathbf{0}$, where $\mathbf{0}$ is the appropriate zero matrix
19. $(\mathbf{A}')' = \mathbf{A}$
20. $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
21. $(k\mathbf{A})' = k\mathbf{A}'$
22. $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

    note: if A is 2×3 and B is 3×4 then AB is 2×4
    ∴ A' is 3×2, B' is 4×3 so $A'B'$ is not defined but $B'A'$ is defined

Cautions:

1. In general, $\mathbf{AB} \neq \mathbf{BA}$
2. If $\mathbf{AB} = \mathbf{AC}$ then it is not always true that $\mathbf{B} = \mathbf{C}$

   eg. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 0 & -1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$
   ($\frac{1}{2}$ class do each side)
   ∴ $AB = AC = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}$
3. If $\mathbf{AB} = \mathbf{0}$ then it is not necessarily true that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$

   eg $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

DAY 3

See multivariate analysis

Variance-Covariance Matrix

Recall: the variance of a variable X is given by $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$s^2 = \frac{1}{n-1} \left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2 \right]$, where $\bar{x}$ is the mean and the x_i are n values of X. (Derivation on back

If we have m variables instead of one and we have n values for each of these m variables, we can record the data as an nxm matrix.

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & 2 \\ 6 & 7 & 4 \\ 9 & 10 & 8 \end{bmatrix}$$

where X_1 = days absent, X_2 = attitude score, and X_3 = experience. There are 5 employees. Here m = 3 (three variables) and n = 5 (five values each).

Then $$\mathbf{A'A} = \begin{bmatrix} 134 & 145 & 105 \\ 145 & 163 & 117 \\ 105 & 117 & 86 \end{bmatrix}$$

Note that this matrix is symmetric (see problem 15). The diagonal elements are sums of squares and the off-diagonal elements are sums of cross-products. e.g. $86 = \Sigma X_3^2$, $117 = \Sigma X_2 X_3$.

See Numbertaking 8/4 - p18 ... 15 ...

Let U be an nxn matrix with each $u_{ij} = 1$ (in the example n = 5). Now calculate $\mathbf{A'UA}$. For the example

Do

$$\mathbf{A'UA} = \begin{bmatrix} 1 & 0 & 4 & 6 & 9 \\ 1 & 2 & 3 & 7 & 10 \\ 1 & 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 20 & 23 & 16 \\ 20 & 23 & 16 \\ 20 & 23 & 16 \\ 20 & 23 & 16 \\ 20 & 23 & 16 \end{bmatrix}$$

Note that each column of the second matrix has the sum of the values of the corresponding variable written n times. e.g. $16 = \Sigma X_3$.

$$= \begin{bmatrix} 400 & 460 & 320 \\ 460 & 529 & 368 \\ 320 & 368 & 256 \end{bmatrix}$$

Note that this matrix is symmetric. Also $460 = (\Sigma X_1)(\Sigma X_2)$, $256 = (\Sigma X_3)^2$, etc.

show 16(1+1+2+4+8) = 16·16

The mean-corrected matrix of sums of squares and cross-products is given by $\mathbf{S} = \mathbf{A'A} - (1/n)\,\underbrace{\mathbf{A'UA}}_{\text{square of sums}}$. Sum of squares

In the example, $$\mathbf{S} = \begin{bmatrix} 134 & 145 & 105 \\ 145 & 163 & 117 \\ 105 & 117 & 86 \end{bmatrix} - (1/5) \begin{bmatrix} 400 & 460 & 320 \\ 460 & 529 & 368 \\ 320 & 368 & 256 \end{bmatrix} = \begin{bmatrix} 54 & 53 & 41 \\ 53 & 57.2 & 43.4 \\ 41 & 43.4 & 34.8 \end{bmatrix}$$

Note that $34.8 = 86 - (1/5)(16)^2 = \Sigma X_3^2 - (1/5)(\Sigma X_3)^2$

The variance-covariance matrix is given by $\mathbf{C} = \frac{1}{n-1} \mathbf{S}$

In the example, $$\mathbf{C} = (1/4)\,\mathbf{S} = \begin{bmatrix} 13.5 & 13.25 & 10.25 \\ 13.25 & 14.3 & 10.85 \\ 10.25 & 10.85 & 8.7 \end{bmatrix}$$

Note that $8.7 = (1/4)[86 - (1/5)(16)^2]$

$= (1/4)[\Sigma X_3^2 - (1/5)(\Sigma X_3)^2]$,

which is the variance of X_3.

13.25 cov(X_1, X_2)

Recall: If a is a real number, then a^{-1} exists provided $a \neq 0$.
So in solving $ax = b$, $x = a^{-1}b$ if a^{-1} exists, i.e. division exists

Comment: many math difficulties arise in grade 4–5 with division.

note $\begin{cases} 0x = 1 \\ 0x = 0 \end{cases}$

QUESTION: Can we do something like this for matrices?
i.e. if $AX = B$ is $X = A^{-1}B$ and what is A^{-1}?

Definition A is a square matrix. The inverse of A, denoted A^{-1}, is a square matrix of the same size as A with the property that $AA^{-1} = A^{-1}A = I$. If such a matrix exists we say A is non-singular or invertible.

EXAMPLES

① $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$. Earlier we checked that $AB = BA = I$.
So $B = A^{-1}$

Pr ② $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

② ③ Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$. Find, if possible, A^{-1}.

Suppose $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. $\begin{bmatrix} 2a+5c & 2b+5d \\ a+2c & b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{cases} 2a+5c = 1 \\ a+2c = 0 \end{cases}$ and $\begin{cases} 2b+5d = 0 \\ b+2d = 1 \end{cases}$

$\begin{cases} a+2c = 0 \\ 2a+5c = 1 \end{cases}$ $\quad$ $\begin{cases} b+2d = 1 \\ 2b+5d = 0 \end{cases}$

$\begin{cases} -2a-4c = 0 \\ 2a+5c = 1 \end{cases}$ $\quad$ $\begin{cases} -2b-4d = -2 \\ 2b+5d = 0 \end{cases}$

$c = 1$ $\quad$ $d = -2$

$a + 2(1) = 0$ $\quad$ $b + 2(-2) = 1$

$a = -2$ $\quad$ $b = 5$

$\therefore A^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$. check: $\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

③ Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$. Find, if possible, A^{-1}

Suppose $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{cases} 2a + 3c = 1 \\ 4a + 6c = 0 \end{cases}$$

$$\begin{cases} 4a + 6c = 2 \\ 4a + 6c = 0 \end{cases}$$

$0 = 2$, which is impossible so A^{-1} does not exist.

④ See exercise #12 for general 2×2 to make up examples like ① ②

See exercise #22 for another example when A^{-1} does not exist.

<u>General Algorithm for Finding A^{-1}</u>

Go back to $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$

We solved $\begin{cases} 2a + 5c = 1 \\ a + 2c = 0 \end{cases}$ and then $\begin{cases} 2b + 5d = 0 \\ b + 2d = 1 \end{cases}$

write $\left[\begin{array}{cc|c} 2 & 5 & 1 \\ 1 & 2 & 0 \end{array}\right]$ and $\left[\begin{array}{cc|c} 2 & 5 & 0 \\ 1 & 2 & 1 \end{array}\right]$

$R_1 \leftrightarrow R_2$ $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 5 & 1 \end{array}\right]$ and $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 5 & 0 \end{array}\right]$

$-2R_1 + R_2$ $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$ and $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \end{array}\right]$

$-2R_2 + R_1$ $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right]$ and $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$

$\therefore \begin{matrix} a = -2 \\ c = 1 \end{matrix}$ and $\begin{matrix} b = 5 \\ d = -2 \end{matrix}$

Do together!! $\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix} \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right]$ A^{-1}

DAY 4

also used for other things we will do

The Gaussian elimination algorithm for finding A^{-1}

- Take the augmented matrix $[A \,\vdots\, I]$.
- Perform elementary row operations
  - $R_i \longleftrightarrow R_j$
  - kR_i
  - $kR_i + R_j$
- Do these EROs until you get $[I \,\vdots\, A^{-1}]$.
- If I can't be obtained then A is not invertible.

A^{-1}
- 2×2 exercise 12
- 3×3 method using determinants

Computers and some calculators can get A^{-1}.

<u>Note</u>: There are other ways to calculate A^{-1} but this is close to what a computer does. Also we will use EROS for other purposes.

<u>EXAMPLES</u>

① $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Find A^{-1}

$3a+5b=1$, $a+2b=0$ $\qquad$ $3c+5d=0$, $c+2d=1$

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{-1R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

② ~~$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$~~

~~$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & 1 \end{array}\right]$ $\therefore A^{-1}$ doesn't exist~~

$A = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 4 & 8 & 0 & 1 \end{array}\right] \xrightarrow{-4R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{4}{3} & 1 \end{array}\right]$$

$\therefore A^{-1}$ does not exist

Class

(3) $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix}$ Find A^{-1}, if it exists

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} -1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1} \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 6 & 6 & 0 & 2 & 1 \end{array}\right] \xrightarrow{-6R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -6 & -6 & 2 & 1 \end{array}\right] \xrightarrow{-\frac{1}{6}R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & -1 & \\ 0 & 1 & 2 & 1 & 0 & \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & \end{array}\right]$$

$$\xrightarrow[R_3+R_1]{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & -\frac{4}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{6} \end{array}\right] \xrightarrow{3R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \frac{2}{3} & \frac{5}{6} \\ 0 & 1 & 0 & -1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{6} \end{array}\right]$$

$$\therefore\ A^{-1} = \begin{bmatrix} -2 & \frac{2}{3} & \frac{5}{6} \\ -1 & \frac{2}{3} & \frac{1}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -12 & 4 & 5 \\ -6 & 4 & 2 \\ 6 & -2 & -1 \end{bmatrix}$$

NOTE: If $AX = B$ and A^{-1} exists then $X = A^{-1}B$

$Ax = b$ $\qquad$ $x = A^{-1}b$

Problem #25-29

Some properties of the inverse

① If a row (a column) of A is all zeros then A^{-1} does not exist. (i.e. 0^{-1} does not exist)

② If a row (a column) of A is a multiple of another row (or column) of A then A^{-1} does not exist.

③ $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots\, d_n \end{bmatrix}$ $\quad D^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \\ & \frac{1}{d_2} & 0 \\ 0 & & \ddots\, \frac{1}{d_n} \end{bmatrix}$ provided each $d_i \neq 0$.

In particular, $\underline{I}^{-1} = \underline{I}$

④ $(AB)^{-1} = B^{-1}A^{-1}$
$\quad \neq A^{-1}B^{-1}$

⑤ $(A')^{-1} = (A^{-1})'$

⑥ $(A^{-1})^{-1} = A$

~~⑦ If A is symmetric and A^{-1} exists then A^{-1} is symmetric.~~

⑦ N.B. $(A+B)^{-1} \neq A^{-1} + B^{-1}$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \cancel{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A^{-1} = A \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$A+B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad A^{-1}+B^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$

$(A+B)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

DAY 5

Def: The rank of a matrix is the number of nonzero rows in the echelon form of the matrix. Write: $r(A)$

eg ① $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ rank is 2 $\quad r(A) = 2$
(Note: A^{-1} does not exist)

② $\begin{bmatrix} 1 & 0 & * & 0 & 0 & * & * \\ 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & k \end{bmatrix}$

If $k = 0$ rank is 4 $r(A)=4$
If $k \neq 0$ rank is 5 $r(A)=5$

② $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 13 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

rank is 2

syllabus

Note: If A is $n \times n$ with rank n (full rank) then A is nonsingular.
If A is $n \times n$ with rank $< n$ then A^{-1} does not exist.

Problem 30

Simultaneous Equations

$$\underset{m\times n}{A}\;\underset{n\times 1}{X} = \underset{m\times 1}{B}$$

m equations
n unknowns

$Ax=b$

Studied in causal models (structural equations), linear models (least squares fit), etc

Recall: H.S. algebra
1 eq, 1 unkn
2 eq, 2 unkn
possibly 3 eq, 3 unkn

If $m \neq n$ $X = A^{-1}B$ is not possible

EXAMPLES (handout)

①
$$3x_1 - 5x_2 = -23$$
$$x_1 + x_2 = 3$$
$$6x_1 + 2x_2 = 2$$

$$\underbrace{\left[\begin{array}{cc|c} 3 & -5 & -23 \\ 1 & 1 & 3 \\ 6 & 2 & 2 \end{array}\right]}_{\text{augmented matrix}} \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -5 & -23 \\ 6 & 2 & 2 \end{array}\right] \xrightarrow[-6R_1+R_3]{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -8 & -32 \\ 0 & -4 & -16 \end{array}\right]$$

coefficient matrix

$$\xrightarrow{-\frac{1}{8}R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & -4 & -16 \end{array}\right] \xrightarrow{+4R_2+R_3} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right] \quad \therefore \; x_1 = -1, \; x_2 = 4$$

eg intersecting three straight lines in a plane

Problem 31 c, 36

note: Third equation is redundant.
(3 row is $\frac{1}{2}$ row 1 + $\frac{9}{2}$ row 2)

②
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

This last equation reads $0a + 0b + 0c = 5$. No solution.
System is overdetermined

See also EXAMPLE 4

Leads to least squares methods (a best approximation to a solution)

Problem 31 a d

(3) "Urban Planning"

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 800 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 1 & 600 \end{array}\right]}_{\text{augmented matrix}} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 800 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & -1 & 0 & 1 & -200 \end{array}\right]$$

(the first four columns: coefficient matrix)

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 800 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 1 & 1 & 500 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 800 \\ 0 & 1 & 1 & 0 & 700 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 800 \\ 0 & 1 & 0 & -1 & 200 \\ 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} \underline{1} & 0 & 0 & 1 & 600 \\ 0 & \underline{1} & 0 & -1 & 200 \\ 0 & 0 & \underline{1} & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note: x_4 is arbitrary and can be given any value (context $\Rightarrow x_4 \geq 0$ and x_4 an integer).

The value is called a parameter; write θ or ϕ or a, etc.

e.g. $x_4 = a$
$x_3 = 500 - a$
$x_2 = 200 + a$
$x_1 = 600 - a$

so note: $a \leq 500$
$x_3 \geq 0$
$x_2 \leq 700$
$x_1 \geq 100$

Prob. 31 be

e.g. $x_4 = 200$, $x_3 = 300$, $x_2 = 400$, $x_1 = 400$

$x_4 = 0$, $x_3 = 500$, $x_2 = 200$, $x_1 = 600$
(i.e. closing this section will not disrupt overall flow. City planners can see which streets are critical for a smooth flow).

Here there are many solutions.

The system is underdetermined.

DAY 6

④ EXAMPLE (Do before summary)

$$\begin{cases} x_1+3x_2+x_3+x_4=3 \\ 2x_1+6x_2+2x_3-x_4=15 \\ 3x_1+10x_2+3x_3+3x_4=9 \\ 4x_1+12x_2+4x_3+4x_4=12 \quad (13) \\ 5x_1+15x_2+5x_3+5x_4=15 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 2 & 6 & 2 & -1 & 15 \\ 3 & 10 & 3 & 3 & 9 \\ 4 & 12 & 4 & 4 & 12 \\ 5 & 15 & 5 & 5 & 15 \end{array}\right] \end{array} (13) \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 & 9 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] (1)$$

leading entry in a row may not be a pivot (dummy)

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] (1) \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] (1) \longrightarrow \text{no soln}$$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} \underline{1} & 0 & 1 & 0 & 6 \\ 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = a$

$x_4 = -3$

$x_2 = 0$

$x_1 = 6 - a$

$m = 4$
$n = 4$
$r = 3$

many soln

⑤

$$[A \vdots B] \to \cdots \to \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \underline{1} & 0 & 0 & 0 & 2 \\ 0 & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 & 1 \\ 0 & 0 & 0 & \underline{1} & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$m = 6$
$n = 4$
$r = 4$

unique soln

⑥

$$[A \vdots B] \to \cdots \to \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \underline{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \underline{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

~~$p \neq 0$ no solution~~

~~$p = 0$ $x_4 = a$, $x_3 = 1 + 2a$, $x_2 = b$, $x_1 = 4 - 2a - 3b$~~

$m = 3$
$n = 4$
rank A = rank $A \vdots B = 2$
$4 - 2 = 2$ fr

$x_4 = a$, $x_3 = -4a$

$x_2 = b$, $x_1 = 2a - 3b$

(I) $AX = B$ $\qquad \underset{m \times n}{A}$ $\qquad$ m eq, n unknown.

$\text{rank } A = \text{rank } [A \vdots B]$?

If no then no solution (ex ②; problem 3|a)

If yes then solution or solutions.

rank $= r$

$r = n$ (ex ①; problem 3|c)

$r < n$ — $n - r$ free variables (ex. ③; problem 3|b)

∴ 3 possibilities

(II) $AX = 0$ $\qquad$ Homogeneous System

Note: $\text{rank } A = \text{rank } [A \vdots 0]$

$r = n$ or $r < n$

one solution $x_1 = \cdots = x_n = 0$ $\qquad$ many solutions (eg. problem 3|b).

∴ 2 possibilities

N.B. number of equations is not so important; rank is.

Problems: 32 − 35

Def. a linear combination of vectors is a sum of scalar multiples of the vectors.

EX Is $\begin{bmatrix}1\\2\\9\end{bmatrix}$ a linear combination of $\begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}$, and $\begin{bmatrix}1\\3\\k\end{bmatrix}$?

$$\begin{bmatrix}1\\2\\9\end{bmatrix} = a\begin{bmatrix}2\\1\\3\end{bmatrix} + b\begin{bmatrix}3\\2\\1\end{bmatrix} + c\begin{bmatrix}1\\3\\k\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\9\end{bmatrix} = \begin{bmatrix}2a+3b+c\\a+2b+3c\\3a+b+kc\end{bmatrix} \quad \therefore \quad \left[\begin{array}{ccc|c}2&3&1&1\\1&2&3&2\\3&1&k&9\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&3&2\\0&-1&-5&-3\\0&-5&k-9&3\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&2&3&2\\0&1&5&3\\0&0&k+16&18\end{array}\right]$$

If $k=-16$ the answer is no

If $k \neq -16$ the answer is yes; e.g. $k=2$ we get $\left[\begin{array}{ccc|c}1&2&3&2\\0&1&5&3\\0&0&18&18\end{array}\right]$

$$\to \left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&-2\\0&0&1&1\end{array}\right]$$

ie $$\begin{bmatrix}1\\2\\9\end{bmatrix} = 3\begin{bmatrix}2\\1\\3\end{bmatrix} - 2\begin{bmatrix}3\\2\\1\end{bmatrix} + 1\begin{bmatrix}1\\3\\2\end{bmatrix}$$

geometrically

Problem #42

Use straws!

Def a collection of vectors $a_1, \dots, a_n$ is <u>linearly dependent</u> if at least one of them can be written as a linear combination of the others.

we don't know which one

i.e. $a_1, \dots, a_n$ is lin. dep. if there exist scalars $k_1, \dots, k_n$ not <u>all</u> zero such that $k_1 a_1 + \dots + k_n a_n = 0$

see Nambooduri p5 for ($\Leftarrow$)

we do ($\Rightarrow$) → e.g. a_1 lin comb of others,

i.e. $k_1 \begin{bmatrix} \; \end{bmatrix} + \dots + k_n \begin{bmatrix} \; \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

i.e. the resulting homogeneous system of equations has a <u>nonzero solution</u>.

i.e. the rank of $\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$ is less than n.

If a collection of vectors is not lin. dep. it is called <u>linearly independent</u>

i.e. the rank of $\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$ equals n (resulting homogeneous system has only zero solution)

Ex Are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ p \end{bmatrix}$ linearly dependent or linearly independent?

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & p \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & p-1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & p+3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & p+3 \end{bmatrix}$$

If $p = -3$ then linearly dependent
if $p \neq -3$ then linearly independent.

$$\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c = 1$
$b = 2$
$a = 1$

$1\begin{bmatrix} \; \end{bmatrix} + 2\begin{bmatrix} \; \end{bmatrix} + 1\begin{bmatrix} \; \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Problem # 43, 44

DAY 7

N.B. REVIEW
- orthogonal $a^t \cdot b = 0$ $a \cdot b = 0$
- length $|a| = \sqrt{a^t a}$

The Normal Equations

Given m independent variables $X_1, X_2, \cdots, X_m$ and the dependent variable Y. We want to find a linear relationship (regression equation) $Y = b_0 + b_1X_1 + b_2X_2 + \cdots + b_mX_m$. Suppose that for each of the variables we have n values (observations). We can then write ther system of equations $\mathbf{X\,B} = \mathbf{Y}$, where

$$\mathbf{X} = \begin{bmatrix} 1 & | & | & & | \\ 1 & | & | & & | \\ \vdots & X_1 & X_2 & \cdots & X_m \\ 1 & | & | & & | \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$n \times (m+1)$ $\qquad$ $n \times 1$ $\qquad$ $(m+1) \times 1$

$X\beta = y$

Recall p 23-24 in Namboodiri

not lin dep.

Y is not a lin comb of X_i ; $X_1, \ldots, X_m, Y$ are lin. indep

Typically the system has no solution (it is overdetermined) so we try to find a matrix $\hat{\mathbf{B}}$ so that $\mathbf{Y} = \mathbf{X}\hat{\mathbf{B}} + \mathbf{E}$, where

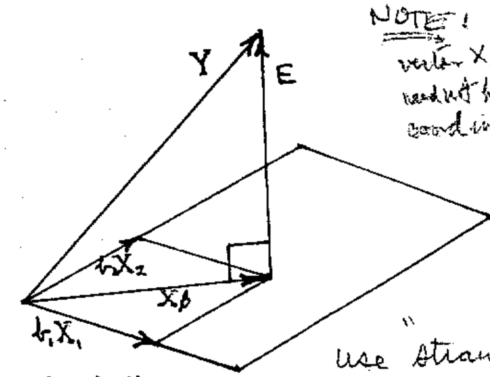
$$\mathbf{E} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

is the error term and $\hat{\mathbf{B}}$ is the best approximation to a solution in the sense that the length of the vector $\mathbf{E}$ is minimized, that is, $\sqrt{\mathbf{E}'\cdot\mathbf{E}} = \sqrt{\Sigma e_i^2}$ is as small as possible. This is the idea of "least squares."

NOTE! vector X_1, X_2, Y need not have coord in 3-space

We will look at a geometric argument to obtain $\hat{\mathbf{B}}$. Recall that the shortest distance from a point to a plane is measured along a perpendicular line from the point to the plane. Also recall that a line will be perpendicular to a plane if it is perpendicular (orthogonal) to every line in the plane passing through the point of intersection of the line and the plane. Using the generalization of these geometric facts, we want to choose $\hat{\mathbf{B}}$ so that the vector $\mathbf{E}$ is orthogonal to the vector $\mathbf{XB}$ for every choice of $\mathbf{B}$. That is, we want to have

i.e. every β



use "straws"

dot

$(\mathbf{X\,B})'\cdot\mathbf{E} = 0$ (vectors are orthogonal when the inner product is 0)
$(\mathbf{X\,B})'\cdot(\mathbf{Y} - \mathbf{X}\hat{\mathbf{B}}) = 0$ (since from above we want $\mathbf{Y} = \mathbf{X}\hat{\mathbf{B}} + \mathbf{E}$)
$(\mathbf{X\,B})'(\mathbf{Y} - \mathbf{X}\hat{\mathbf{B}}) = 0$ (writing the inner product as matrix multiplication)
$(\mathbf{B}'\mathbf{X}')(\mathbf{Y} - \mathbf{X}\hat{\mathbf{B}}) = 0$ (a property of the transpose of a matrix) Problem #13 d
$\mathbf{B}'[\mathbf{X}'(\mathbf{Y} - \mathbf{X}\hat{\mathbf{B}})] = 0$ (a property of matrix multiplication)
$\mathbf{B}'[\mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}\hat{\mathbf{B}}] = 0$ (a property of matrix multiplication)

Since this last statement must be true for <u>every</u> choice of $\mathbf{B}$, we must have $\mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}\hat{\mathbf{B}} = \mathbf{0}$.
This gives us the <u>normal equations</u>

$$\boxed{\mathbf{X}'\mathbf{X}\hat{\mathbf{B}} = \mathbf{X}'\mathbf{Y}}$$

Note that if $\mathbf{X}'\mathbf{X}$ is non-singular (invertible) then $\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.
Even if it is invertible, we can avoid calculating the inverse by solving the system of equations directly, i.e.

$$[\mathbf{X}'\mathbf{X} \mid \mathbf{X}'\mathbf{Y}] \longrightarrow \cdots \longrightarrow [\mathbf{I} \mid \hat{\mathbf{B}}]$$

Ex ① $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $X = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $[1 \cdots 1]\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\hat{\beta} = [1 \cdots 1]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \Rightarrow n\hat{\beta} = \Sigma y \Rightarrow \hat{\beta} = \frac{1}{n}\Sigma y_i$
(sample mean)

② $X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$ see p 41-44 in Namboodiri NB $E \perp X$

③ over

2'

EXAMPLE

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 0 \\ 4 & 4 & -4 \end{bmatrix} \qquad \text{Find } \det A$$

(a) Expansion along 2nd row

$$\det A = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$= 2(-1)^{2+1} \det A_{21} + (-2)(-1)^{2+2} \det A_{22} + 0\,(-1)^{2+3} \det A_{23}$$

$$= 2(-1) \begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2(+1) \begin{vmatrix} 1 & 1 \\ 4 & -4 \end{vmatrix} + 0(-1) \begin{vmatrix} 1 & -1 \\ 4 & 4 \end{vmatrix}$$

$$= -2(4-4) - 2(-4-4) - 0(4+4)$$

$$= -2(0) - 2(-8) - 0(8) = 16$$

(b) Expansion down 3rd column

$$\det A = a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33}$$

$$= 1(-1)^4 \begin{vmatrix} 2 & -2 \\ 4 & 4 \end{vmatrix} + 0(-1)^5 \begin{vmatrix} 1 & -1 \\ 4 & 4 \end{vmatrix} + (-4)(-1)^6 \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}$$

$$= 1(8+8) + 0 + 0 = 16$$

Properties (Reverse side of handout)

② $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{-3R_1+R_2} B = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

④ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{3R_1} B = \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$

EXAMPLE

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 0 \\ 4 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 8 & -8 \end{bmatrix} \xrightarrow{-} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 8 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore \det A = -(1)(8)(-2) = 16.$$

DAY 8

① Scatterplot
② Style

motivating
EXAMPLE

$$A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

(a) $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \therefore AX = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

(b) $X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \therefore AX = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

one color

Draw

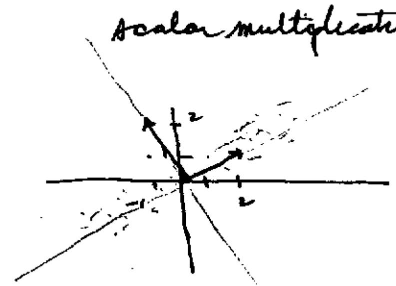
(c) $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \therefore AX = \begin{bmatrix} 18 \\ 9 \end{bmatrix} = 9\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(d) $X = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \therefore AX = \begin{bmatrix} -4 \\ 8 \end{bmatrix} = 4\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

another color

note: In (c) and (d) the matrix multiplication turns out to be scalar multiplication.

not here



scatterplot (later)

Def: · eigenvalues of A (char. values, latent values, etc.)
· eigenvectors of A

$\boxed{AX = \lambda X \text{ for some } X \neq 0}$

How to find.

① $AX = \lambda X$ $\quad A \; n\times n$
$AX - \lambda X = 0$
$AX - \lambda I X = 0$
$(A - \lambda I)X = 0$
$X \neq 0 \Rightarrow A - \lambda I$ has rank $< n$
$\Leftrightarrow \det(A - \lambda I) = 0$ characteristic equation

② Solve $(A - \lambda I)X = 0$ to get X

HANDOUT

Do example

~~$\lambda = 9 \quad \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} 2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 1 \end{bmatrix}$~~

~~$\lambda = 4 \quad \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad k\begin{bmatrix} -1 \\ 2 \end{bmatrix}$~~

Ex: $A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = \lambda^2 - 13\lambda + 36 = (\lambda-9)(\lambda-4)$$

$\therefore \lambda = 9, 4$ are the eigenvalues

Eigenvectors:

$\lambda = 9$ $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (or any scalar multiple)

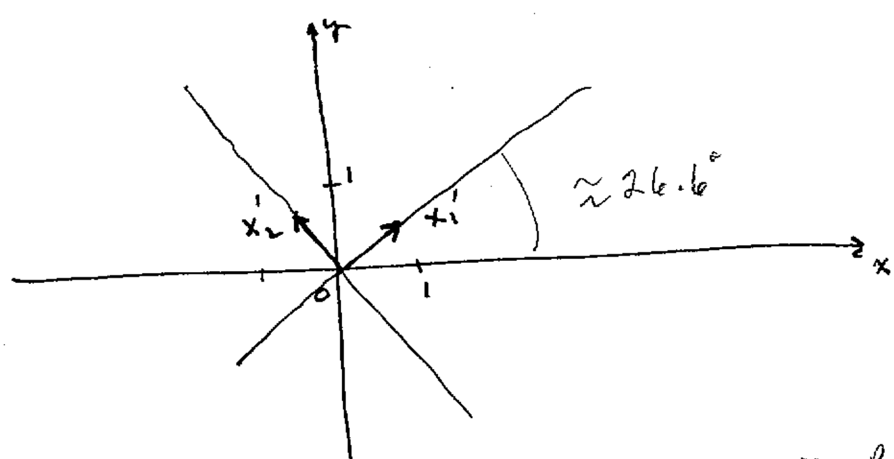
$\lambda = 4$ $\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (or any scalar mult.)

Note: the eigenvectors are linearly independent
" " " orthogonal
trace A = sum of the eigenvalues (total variance).
We can obtain unit vectors in the direction of the eigenvectors (see prob #8h)

$\lambda = 9$ $\quad X_1' = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$; $\quad \lambda = 4$ $\quad X_2' = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

Geometric interpretation

$X_1' \approx \begin{bmatrix} .89 \\ .45 \end{bmatrix}, \quad X_2' \approx \begin{bmatrix} -.45 \\ .89 \end{bmatrix}$



Note $A = P \Lambda P^{-1} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

note: P is an orthogonal matrix (see Exam #25)
so $P^{-1} = P'$

sometimes correlation matrix used
or mean-corrected sums of squares S on p14

Suppose $A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$ is the covariance-variance matrix of X_1, X_2

see Noble-Daniel p421
Tucker p 489

Trace $A = 8+5 = 13$ (total variance)

Eigenvalues are 9 and 4

Sum of eigenvals is also 13.

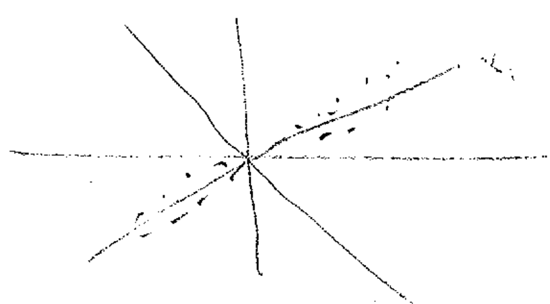
The largest eigenvalue is 9

$\frac{9}{13} \approx .69$ (almost 70% of the total variance

Consider the following linear combination of the variables X_1, X_2 (original variables)

$$Y_1 = \frac{2}{\sqrt{5}} X_1 + \frac{1}{\sqrt{5}} X_2$$

Recall the eigenvector $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ assoc. with $\lambda = 9$.



Covariance near 0 : not correlated
+ : correlated
− : opposite

$B = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$

101 revisited

Types of Functions

Ⓐ Linear $\quad y = mx + b \quad$ slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $\quad m > 0$, $m = 0$, $m < 0$

$b =$ y-intercept

Note: m is constant.

e.g. $\quad y = 2{,}527x + 111{,}375 \qquad x = 0$ corresponds to 1970

Linear Regression best fit

$y =$ number (X1000) of U.S. workers fully insured for Social Security benefits

From Soc. Sec. adm.

Do CPI or Fuelwood

~~Year~~	~~1970~~	~~1975~~	~~1980~~	~~1985~~	~~1990~~	~~1995~~	~~2000~~
~~X~~	~~0~~	~~5~~	~~10~~	~~15~~	~~20~~	~~25~~	~~30~~
~~y~~	~~108,343~~	~~122,160~~	~~140,382~~	~~156,808~~	~~164,011~~	~~173,172~~	~~185,064~~

$y = 0.97x + 32.1 \qquad x = 0$ is 1980

$y =$ consumption in million cu. meters per year.

more generally, $\quad y = b_0 + b_1 X_1 + b_2 X_2 + \cdots + b_n X_n$

Ⓑ Nonlinear

① Quadratic $\quad$ e.g. U.S. military spending - 1980 – 1998 $\;>$ polynomials

② Cubic $\quad$ e.g. AIDS cases in U.S. 1984 – 2002

③ Rational $\quad$ e.g. "Cube Law"

④ Exponential $\quad$ e.g. Prisoners per 100,000 residents in U.S. 1970 – ~~[illegible]~~ 2000

⑤ Power $\quad$ e.g. Zipf's Law (urban concentration, word frequency, etc.).

⑥ Trigonometric $\quad$ e.g. amounts of daylight in A^2, where births each hour,

[Extra time next week].

$y = b_0 + b_1 x + b_2 x^2$

$b_0 + b_1 x_1 + b_2 x_1^2 = y_1$

$X = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \quad \hat{\beta} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ \vdots \end{bmatrix}$

$(X'X)\hat{\beta} = X'Y$ to get parabola

Average Rate of Change of a Function.

N.B. For a linear function, the rate is constant i.e. $m = \frac{y_2 - y_1}{x_2 - x_1}$

1. Life expectancy of women in Finland in 1963 was 72.7
and in 2001 was 81.2

$\therefore$ Average rate of change of life expectancy is $\frac{81.2 - 72.7}{2001 - 1963} = \frac{8.5}{38} \approx 0.22$

i.e. about 0.22 ~~years~~ per year.

2. Gallup Poll: Do you approve (or disapprove) of the way George W. Bush is handling his job as president?

change

Date	% approve
May 2001	57
Oct. 2001	88
Oct. 2002	61
July 2003	57%

Date	% approve
May 2001	57
Oct 2001	88
Oct. 2004	48
~~May~~ 2005	48

(a) ↑6.2% per mo
(b) (b) −.2% per mo
(c) (c) no change

(a) May 2001 to Oct. 2001 $\frac{88-57}{5} \doteq 6.2\%$ per month increase

(b) Oct. 2001 to Oct 2002 $\frac{61-88}{12} = -2.25\%$ " " decrease

(c) May 2001 to July 2003 $\frac{57-57}{26} = 0\%$ " "

(d) May 2001 to Oct. 2002 $\frac{61-57}{17} \approx 0.24\%$ " " increase

Note: Not constant; some increase, some decrease, some no change
increase in (a) is greater than increase in (d).

Eminent social scientist Yogi Berra once noted the trickiness in tracking systematic changes over time
"The future ain't what it used to be!"

3. World Population

4 3 $y = f(x) = 9 + 8x - x^2$ (Rather easier to work with than EXAMPLE 1.)

Find the average rate of change of y as x changes from

(a) 2 to 5

$$\frac{f(5) - f(2)}{5-2} = \frac{24 - 21}{5-2} = \frac{3}{3} = 1 \quad \text{(increase)}$$

(b) 2 to 7

$$\frac{f(7) - f(2)}{7-2} = \frac{16-21}{5} = \frac{-5}{5} = -1 \quad \text{(decrease)}$$

(c) 2 to 6

$$\frac{f(6) - f(2)}{6-2} = \frac{21-21}{4} = \frac{0}{4} = 0 \quad \text{(no change)}$$

Notation: $\frac{\Delta y}{\Delta x}$ for average rate of change

(a) $\frac{\Delta y}{\Delta x} = \frac{f(2+3) - f(2)}{3}$ (b) $\frac{\Delta y}{\Delta x} = \frac{f(2+5) - f(2)}{5}$ (c) $\frac{\Delta y}{\Delta x} = \frac{f(2+4}{4}$

In general, if $y = f(x)$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

"difference quotient"

"average rate of change of y as x changes from x to $x + \Delta x$"

Graphically, this represents the slope of the secant line joining the two points $(x, f(x))$ and $(x+\Delta x, f(x+\Delta x))$

"Physics" example (everyone has experienced).

Enter toll road at mile 36 at 11:30 ; exit at mile 96 at 12:30

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\Delta s}{\Delta t} = \frac{96-36}{12{:}30-11{:}30} = \frac{60}{1} = 60 \text{ m.p.h.}$$

Do quickly!

or $\dfrac{1 \text{ mile}}{\frac{1}{60} \text{hr} = 1 \text{ min}}$ = 60 mph average vel.

$\dfrac{\frac{1}{60} \text{mi} = 88'}{\frac{1}{3600} \text{hr} = 1 \text{ sec}}$ = 60 mph average vel.

$\dfrac{.88' = 10.56''}{.01 \text{ sec}}$ = 60 mph average vel.

$\dfrac{1.056''}{.001 \text{ sec}}$ = 60 mph ~~average~~ vel.

etc.

Yogi

"The future ain't what it used to be"

Change over time

DAY 1 00

Question what is the instantaneous velocity? "snapshot of speedometer at an instant."

(at be some value or not moving!

Relate to $y = f(x) = 9+8x-x^2$ and take $x = 2$ fixed

$\Delta x = 5$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(2+5)-f(2)}{5} = -1$

$\Delta x = 4$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(2+4)-f(2)}{4} = 0$

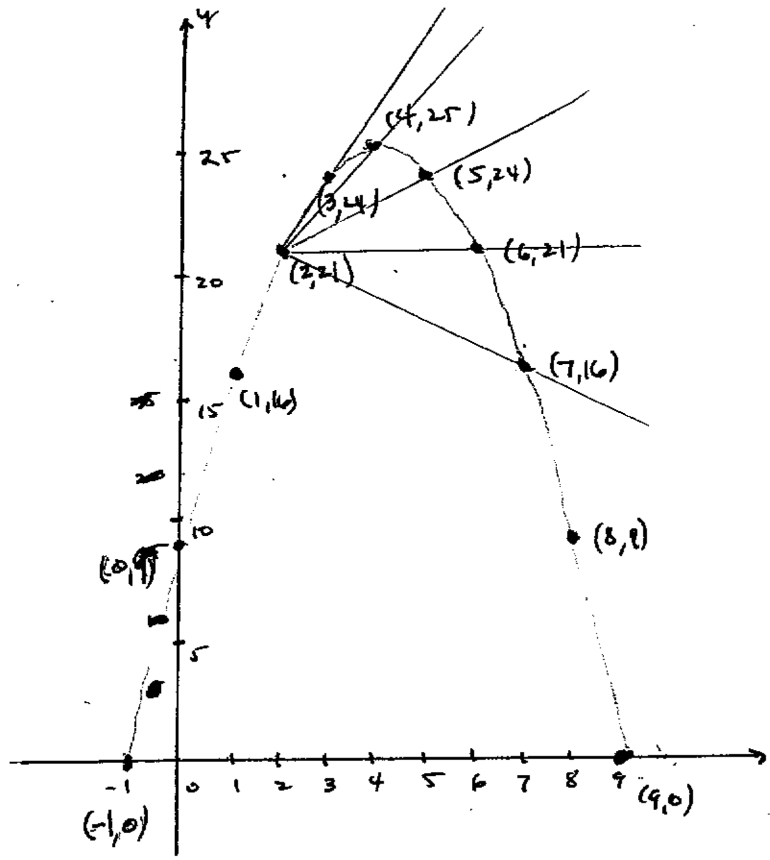
$\Delta x = 3$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(2+3)-f(2)}{3} = 1$

$\Delta x = 2$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(2+2)-f(2)}{2} = \dfrac{f(4)-f(2)}{2} = \dfrac{25-21}{2} = 2$

$\Delta x = 1$ $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(2+1)-f(2)}{1} = \dfrac{f(3)-f(2)}{1} = \dfrac{24-21}{1} = 3$

Note: we can't let $\Delta x = 0$. Why?

5



- Sketch on board
- straw or stick

$$y = f(x) = 9 + 8x - x^2 \qquad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{f(2+\Delta x) - f(2)}{\Delta x}$$

$x = 2$ fixed

(a) $\Delta x = 5 \qquad \dfrac{\Delta y}{\Delta x} = \dfrac{f(7) - f(2)}{7-2} = \dfrac{16-21}{5} = -1$

(b) $\Delta x = 4 \qquad \dfrac{\Delta y}{\Delta x} = \dfrac{f(6) - 21}{4} = \dfrac{21-21}{4} = 0$

(c) $\Delta x = 3 \qquad \dfrac{\Delta y}{\Delta x} = \dfrac{f(5) - 21}{3} = \dfrac{24-21}{3} = 1$

(d) $\Delta x = 2 \qquad \dfrac{\Delta y}{\Delta x} = \dfrac{f(4) - 21)}{2} = \dfrac{25-21}{2} = 2$

(e) $\Delta x = 1 \qquad \dfrac{\Delta y}{\Delta x} = \dfrac{f(3) - 21}{1} = \dfrac{24-21}{1} = 3$ $\qquad y = 3x + 15$ is secant line at $x = 2$

N.B. $\Delta x = 0$ is impossible

(a) — (e) give slopes of secant lines

Question: what is the slope of the line tangent at $x = 2$?

[Two points determine a line! Can we do it with one point?]

point and radius

Lets go back and let

$\Delta x = .5 \quad \frac{\Delta y}{\Delta x} = \frac{f(2+.5)-f(2)}{.5} = \frac{f(2.5)-f(2)}{.5} = \frac{22.75-21}{.5} = 3.5$

$\Delta x = .25 \quad \frac{\Delta y}{\Delta x} = \frac{f(2.25)-f(2)}{.25} = \frac{21.9375-21}{.25} = 3.75$

$\Delta x = .1 \quad \frac{\Delta y}{\Delta x} = \frac{f(2.1)-f(2)}{.1} = \frac{21.39-21}{.1} = 3.9$

$\Delta x = .01 \quad \frac{\Delta y}{\Delta x} = 3.99$

$\Delta x = .001 \quad \frac{\Delta y}{\Delta x} = 3.999$

note: in each case we are getting the slope of a secant line, i.e. a line through 2 points and $\frac{\Delta y}{\Delta x}$ is the average rate of change.

what happens if we continue to let Δx get smaller and smaller?
i.e. Δx closer and closer to 0 but not equal to zero? Guess!

i.e. look at $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ as Δx gets "smaller" but not 0.

We call that idea a limit and write $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

it will give the slope of the tangent line (or instantaneous rate of change)

Def: This limit is called the derivative of $y=f(x)$ [marginal]

and written $\frac{dy}{dx}$ or y' or $f'(x)$

Tangent line at (2,21)
$y = 4x + 13$

EXAMPLE: $f(x) = 9+8x-x^2$

$f(x+\Delta x) = 9+8(x+\Delta x)-(x+\Delta x)^2 = 9+8x+8\Delta x - x^2 - 2x\Delta x - \Delta x^2$

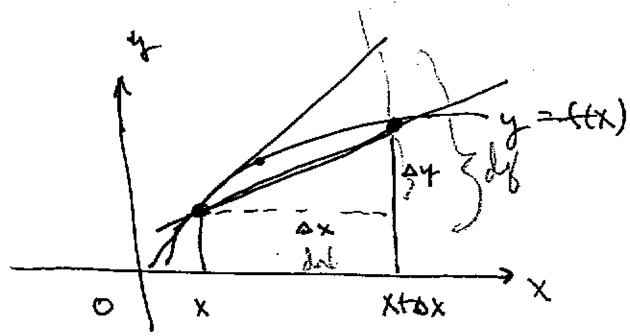
$f(x+\Delta x) - f(x) = 8\Delta x - 2x\Delta x - \Delta x^2$

$\frac{\Delta y}{\Delta x} = \frac{8\Delta x - 2x\Delta x - \Delta x^2}{\Delta x} = 8 - 2x - \Delta x$ since $\Delta x \neq 0$

$\therefore \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 8 - 2x = \frac{dy}{dx} = f'(x)$

note when $x = 2$, $\frac{dy}{dx} = 4$ as we guessed.

SUMMARY



$\frac{\Delta y}{\Delta x}$ = average rate of change
= slope secant line

$\frac{dy}{dx}$ = instantaneous rate of change
= slope tangent line

MATHEMATICAL MODEL

comment: In social science applications, data is usually discrete but we consider it continuous so we can use calculus. (e.g. ~~Clinton~~ Bush approval rating, AIDS cases ~~deaths~~, ~~etc.~~) ~~national Defense Expenditures, etc.~~ U.S. military spending, etc.

Question: Is it possible to find derivatives (i.e. do differentiation) without the long process illustrated in the example? — idea important though!

Fortunately we have rules (theorems) that have been found.

① $y = f(x) = c$ (constant) $\quad \frac{dy}{dx} = f'(x) = 0$

② $y = f(x) = x^n \quad \frac{dy}{dx} = f'(x) = nx^{n-1}$

③ $y = c\,f(x) \quad \frac{dy}{dx} = c\,f'(x)$

④ $y = u(x) \pm v(x) \quad \frac{dy}{dx} = u'(x) \pm v'(x)$

---

other page

EXAMPLES:

① $y = mx + b \quad \frac{dy}{dx} = m \cdot 1 + 0 = m$ fortunately!

② $y = 9 + 8x - x^2 \quad \frac{dy}{dx} = 0 + 8 \cdot 1 - 2 \cdot x = 8 - 2x$ as before!

③ any polynomial: AIDS example ~~$y = -265x^3 + 6220x^2 - 9300x + 265{,}268$~~

~~$\frac{dy}{dx} = -795x^2 + 12440x - 9300$~~

~~$f(3) = 286{,}193 \quad f'(3) = 20{,}865$~~

~~$\therefore f(4) \approx f(3) + f'(3) = 307{,}058$~~

~~from eq., $f(4) = 310{,}628$~~

$f(x) = y = -134x^3 + 7691x^2 - 21112x + 8883$

$f'(x) = \frac{dy}{dx} = -402x^2 + 15382x - 21112$

$f(5) = 78{,}848$

$f'(5) = 45{,}748$

$\therefore f(6) \approx f(5) + f'(5) = 124{,}596$

$f(6) = 130{,}143$ from $f(x)$

Data: 127,238

Chg OVA

④ $y = \frac{7}{\sqrt[4]{x^3}} = \frac{7}{x^{\frac{3}{4}}} = 7x^{-\frac{3}{4}} \quad \therefore \frac{dy}{dx} = -\frac{21}{4}x^{-\frac{7}{4}}$

Deg: '91, '92, '93, '94, '95, '96, '97, '98, '99, '00

EXAMPLES

① $y = f(x) = mx + b \quad f'(x) = \frac{dy}{dx} = m \cdot 1 + 0 = m$ as expected!

② $y = f(x) = 9 + 8x - x^2 \quad f'(x) = \frac{dy}{dx} = 8 - 2x$ as before!

③ $y = f(x) = -0.24x^3 + 6.91x^2 + 1.73x + 9.58$ [AIDS example]

change other sheet

~~$\frac{dy}{dx} = f'(x) = -0.72x^2 + 13.82x + 1.73$~~

~~eg. In 1994 $(x = 10)$~~

~~$f(10) \approx 478$ (law - 494)~~

~~$f'(10) \approx 68$~~

~~$\therefore f(11) \approx 478 + 68 = 546$~~

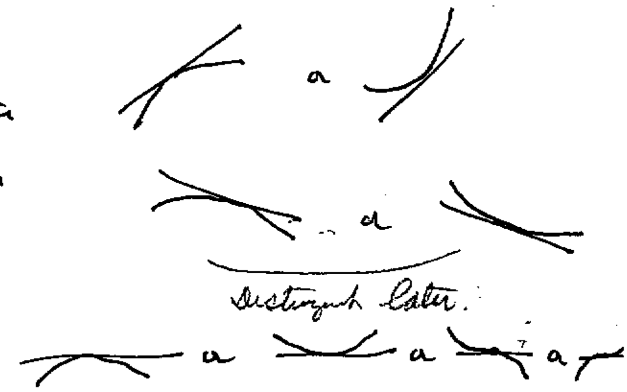
~~(1995 report is 564)~~

④ $y = f(x) = \frac{7}{\sqrt[4]{x^3}} = \frac{7}{x^{\frac{3}{4}}} = 7x^{-\frac{3}{4}}$

$\therefore \frac{dy}{dx} = f'(x) = 7\left(-\frac{3}{4}\right) x^{-\frac{7}{4}} = -\frac{21}{4} x^{-\frac{7}{4}}$

DAY 2

Additional Information from the Derivative



If $f'(a) > 0$ then f is increasing at $x = a$

If $f'(a) < 0$ then f is decreasing at $x = a$

If $f'(a) = 0$ then f is stationary at $x = a$

EXAMPLES

① $f(x) = 9 + 8x - x^2$

$f'(x) = 8 - 2x = 2(4 - x)$

$f'(1) > 0$ so f is increasing at $x = 1$

$f'(x) < 0$ if $x > 4$ so f is decreasing.

$f'(4) = 0$ so f is stationary

② $f(x) = x^3 - 9x^2 - 48x + 52$

$f'(x) = 3x^2 - 18x - 48 = 3(x^2 - 6x - 16) = 3(x - 8)(x + 2)$

+   −   +

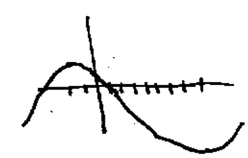
−2   8

incr   decr   incr

GRAPH

$f(-2) = 104$

$f(8) = -396$



③ $f(x) = x^5 + 3$

$f'(x) = 5x^4$. Stationary at $x = 0$. increasing for all other x.

Some more Theorem to get Derivative

⑤ Product Rule

$$y = f(x) = u(x) \cdot v(x)$$

Note: $y = x^2 \cdot x^3$

$$\frac{dy}{dx} = f'(x) = u \frac{dv}{dx} + v \frac{du}{dx}$$
$$= u v' + v u'$$

EXAMPLE: $f(x) = (x^4+7x^2-5)(x^5+3x^3+x-2)$

$$f'(x) = (x^4+7x^2-5)(5x^4+9x^2+1) + (x^5+3x^3+x-2)(4x^3+14x)$$

Note: here we could have multiplied out but later problems we can't.

eg $f(x) = x^2 e^x$

⑥ Quotient Rule

$$y = f(x) = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = f'(x) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} = \frac{v u' - u v'}{v^2}$$

TYPO F 706 p107 correct in F/94 p102

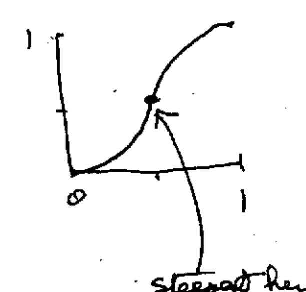
EXAMPLE: (Cube Law)

x = prop. votes
y = prop. seats

$$\text{eg } y = f(x) = \frac{x^3}{(1-x)^3+x^3} = \frac{x^3}{3x^2-3x+1}$$

$$\frac{dy}{dx} = f'(x) = \frac{(3x^2-3x+1)(3x^2) - x^3(6x-3)}{(3x^2-3x+1)^2} = \frac{3x^2(x-1)^2}{(3x^2-3x+1)^2}$$

Note: $f'(.25) = .55\cdot)$
$f'(.4) = 2.2$
$f'(.5) = 3$
$f'(.6) = 2.2$



⑦ Chain Rule.

If f depends on u and u depends on x → f depends on x.

$$\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}$$

$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

EXAMPLES

① $y = f(x) = (3x+1)^2$ or $f(x) = (3x+1)^{20}$ → $g(x) = u^{20}$ $u = 3x+$

$\frac{dy}{dx} = 2(3x+1) \cdot 3$ $\qquad f'(x) = 20(3x+1)^{19} \cdot 3$

strategy, write down

② $y = \sqrt[3]{x^2+1} = (x^2+1)^{\frac{1}{3}}$ $\qquad \frac{dy}{dx} = \frac{1}{3}(x^2+1)^{-\frac{2}{3}} \cdot 2x$

③ $y = f(x) = \frac{1}{(1-x^2)^4} = (1-x^2)^{-4}$ $\qquad \frac{dy}{dx} = -4(1-x^2)^{-5}(-2x) = \frac{8x}{(1-x^2)^5}$

DAY 3.

Recall: $y = f(x)$: gives points on the curve or value of the function

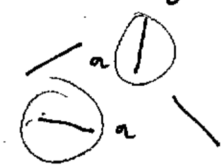
$f'(x)$: gives slope of tangent or measure of how f is changing.

$y = g(x)$

$g'(x)$ how g changes.

Notation: $g'(x)$ would measure how g is changing.

which has larger slope?



Second Derivative

$$y = f(x) = x^4 - 4x^3 + 20$$

$$\frac{dy}{dx} = f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = 12x^2 - 24x = 12x(x-2).$$

Interpretation: Note: f'' measures how f' is changing! [this interpretation not in Quick Calculus]

$f''(a) > 0 \Rightarrow f'$ is increasing at a $\Rightarrow f$ is concave up at a ("holds water)

Do with "straw"

$f''(a) < 0 \Rightarrow f'$ is decreasing at a $\Rightarrow f$ is concave down at a

$f''(a) = 0$ often (but not always) means there is a change in concavity, i.e. an inflection point at $x = a$

EXAMPLES

① $y = f(x) = x^3 - 9x^2 - 48x + 52$

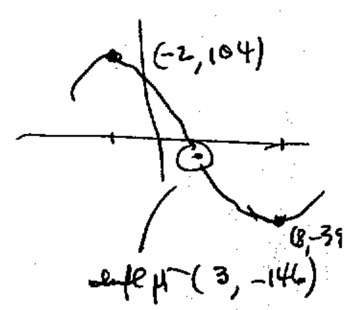
$f'(x) = 3x^2 - 18x - 48 = 3(x+2)(x-8)$

inc -2 dec 8 inc

$f''(x) = 6x - 18 = 6(x-3)$

$-$ $+$

conc do 3 conc up



(2) $y = f(x) = x^4 - 4x^3 + 20$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ — dec. 0 dec 3 inc.

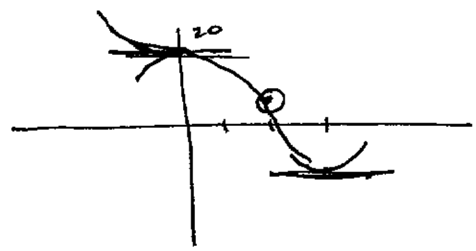
$f''(x) = 12x^2 - 24x = 12x(x-2)$ — + − + ; up 0 down 2 up

e.g. at $x=-1$ decreasing and concave up

at $x=1$ decreasing and concave down

at $x=3$ stationary and concave up.

note: there is an inflection pt at $x=0$ and at $x=2$.



$f(0) = 20$
$f(2) = 4$
$f(3) = -7$

(3) $f(x) = (\cancel{3x+1})^{4} \; (3x+1)^6$

$f'(x) = 6(3x+1)^5 \cdot 3 = 18(3x+1)^5$

$f''(x) = 18 \cdot 5(3x+1)^4 \cdot 3 = 270(3x+1)^4$

note: $f''(-\frac{1}{3}) = 0$

f': − + ; dec $-\frac{1}{3}$ inc — maybe ~~no infl pt~~

f'': + + ; up $-\frac{1}{3}$ up

$f''(-1) > 0$

$f''(0) > 0$

∴ no inflection pt at $x = -\frac{1}{3}$. f is always concave up

Problem 11

(4) "World population grows at rate of increase slow"

(Note) f' = velocity, f'' = acceleration) sometimes used in soc. sci. lit. even though far from physical science

(5) Defense spending (see p 12')

(6) Let $x(t)$ = yearly amount of amount spent by X country (some standard monetary unit).

$y(t)$ = " " " " " " Y country

t = year.

"Mutual Fear Model"

The more one country arms, the more the other is spurred to arm.

i.e. $\frac{dx}{dt} = ay$, $\frac{dy}{dt} = bx$ a, b pos. constants.

i) Since $a, b > 0$, $x, y > 0$, x and y are increasing functions of time.

ii) $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(ay) = a\frac{dy}{dt} = abx > 0$ i.e. $\frac{d^2x}{dt^2} > 0$

so x is concave up.

Similarly y is concave up.

⇒ Runaway Arms Race! (Richardson 1935)

DAY 4 00

changes: own limits, grievances or goodwill

World population grows as rate of increase slows

WASHINGTON (AP)—The world population continues to grow, especially in Africa, although the global rate of growth has slowed in the last five-year period, the U.S. Census Bureau reported yesterday.

Worldwide, the report states, "the trend in the global population growth rate has been generally downward in recent years, the growth rate having declined from about 2.1 per cent in 1965-70 to 1.7 per cent in 1975-79."

DURING THE SAME period, the growth rate for Africa increased from 2.5 per cent to 2.9 per cent, highest in the world, the report disclosed. By comparison, Latin America had a 2.4 per cent growth rate in the 1975-79 period, while the rate was 1.9 per cent in Asia, 1.1 per cent in Oceania, 0.8 per cent in North America and 0.6 per cent in Europe. Oceania includes Australia and islands in the South Pacific.

Earlier, in 1965-70, the growth rate in Latin America was 2.7 per cent; that of Asia was 2.5 per cent; Oceania, 2 per cent; North America, 1.1 per cent; and Europe, 0.8 per cent.

The report notes that, while population growth rates have declined in most parts of the world, the absolute number of people being added yearly to the world's total has not declined. That is because the lower growth rate is applied to an ever-larger base population.

THE REPORT FOCUSES on the 87 countries with populations of five million or more. These countries contain 97 per cent of the world's population.

The world's most populous nation, the report finds, is mainland China with more than one billion people. However, the authors stress there are still uncertainties about the quality and accuracy of population information from China.

India was ranked second with 667 million. The Soviet Union was third with 263 million followed by the United States at 221 million and Indonesia at 148 million.

The continued high growth rate in Africa is occuring "primarily because birthrates are not declining in Africa as they are in other places. Because the birthrates are more or less steady and death rates are declining, that means an increase in the population growth rate," explained Ellen Jamison, chief of the bureau's International Demographic Analysis Branch.

She noted that "in many places in Latin America and Asia there is much wider use of family planning methods than in Africa."

"Also, in some African countries, fertility will increase before it decreases because of improving health conditions," Jamison added.

$\frac{dP}{dt} > 0$

$\frac{d^2P}{dt^2} < 0$

Maxima and Minima (local – "relative" in some books).

1. Find stationary pts, i.e. soln $f'(x)=0$ for x [$f'(x)$ does not exist]
2. Test these values of x either by $f''(a)$ or $\begin{matrix} f'(a^-) \\ f'(a^+) \end{matrix}$

2nd deriv test — 1st deriv test

EXAMPLES

① $f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$

$\therefore x = 1, 3$ are stat. pts.

"Bucket on Floor" (do previous day)

$f''(x) = 6x - 12 = 6(x-2)$

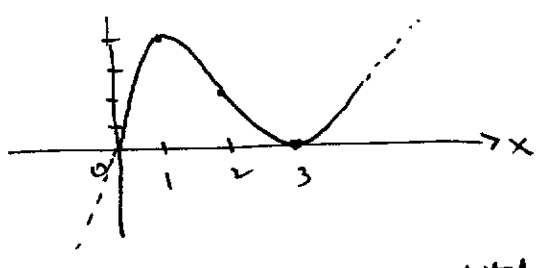
$f''(1) < 0$ $\frown$ $\therefore$ loc. max

$f''(3) > 0$ $\smile$ $\therefore$ loc. min.

note: at $x=2$ we possibly have an I.P.

$f''(1) < 0$

$f''(2) > 0$ $\therefore$ I.P at $x = 2$



$f(1) = 4$

$f(3) = 0$

$f(2) = 2$

why dotted part?

② $f(x) = x^3 + 3$

$f'(x) = 3x^2$ stat. pt at $x = 0$

$f''(x) = 6x$ $f''(0) = 0$ so no info abt max min. $\begin{matrix} f'(-1) > 0 \\ f'(1) > 0 \end{matrix}$ $\therefore$ no max min

Possible I.P. at $x = 0$

$f''(-1) < 0$

$f''(1) > 0$ $\therefore$ I.P. at $x = 0$

③ $f(x) = (2x-1)^4$

$f'(x) = 4(2x-1)^3 \cdot 2 = 8(2x-1)^3$ stat. pt at $x = \frac{1}{2}$

$f''(x) = 24(2x-1)^2 \cdot 2 = 48(2x-1)^2$ $f''(\frac{1}{2}) = 0$ so no info abt max – min

$f'(0) < 0$

$f'(1) > 0$ $\smile$ $\therefore$ loc. min at $x = \frac{1}{2}$

note: even though $f''(\frac{1}{2}) = 0$

$f''(0) > 0$ and $f''(1) > 0$ so no infl. pt at $x = \frac{1}{2}$

4. Profit = Revenue − Cost ("lemonade stand")

$P(x) = R(x) - C(x)$, where x = # items produced.

maximum profit for P?

$$P'(x) = R'(x) - C'(x)$$
$$P'(x) = 0 \Rightarrow R'(x) = C'(x)$$

$\therefore$ maximum profit when marginal revenue = marginal cost.

M.C. = additional cost incurred due to producing one more unit

[at low levels of production, MC decreases as production increases because of experience and efficiencies. at high levels of production, MC increases as production increases because of age of equipment, overtime pay, rise in cost of materials, etc.]

N.B →

TYPOS
p42 F80 $10^0=1$, $10^{-1}=\frac{1}{10}$
p48 F95 The numerical value of $\log e$ is 0.434
so $\ln x = 2.303\ldots \log x$

Exponential Functions

Def: $y = b^x$, $b \neq 1$, $b > 0$

note: $y_1 = b^{x_1}$, $y_2 = b^{x_2}$

$$\frac{y_2}{y_1} = \frac{b^{x_2}}{b^{x_1}} = b^{x_2 - x_1}$$

eg. $y = 2^x$ (not same as $y = x^2$)

$y = \left(\frac{1}{2}\right)^x = 2^{-x}$

$y = 10^x$

$y = e^x$ $\quad e \approx 2.718\ldots$

$P(t) = \cancel{88.5}\ 85.5\,(1.06)^t$ $\quad$ EXAMPLE 4

$t=0$ corresponds to 1970
P = # sentenced prisoners in U.S. (Federal + State) per 100,000 residents

need a calculator (are in HW).

t	P(model)	P(actual)	
0 (1970)	85.5	96	(low)
10 (1980)	153.12	139	(high)
15 (1985)	204.91	202	
20 (1990)	~~279.9~~ 274.21	297	(low)
30 (2000)	~~[illegible]~~ 491.07	469	(high)
33 2003	584.87	482	

Graphs $\quad y = 2^x$, $y = 2^{-x}$

Properties
(1) $b^{x_1} b^{x_2} = b^{x_1 + x_2}$
(2) $b^{x_1} \div b^{x_2} = b^{x_2 - x_1}$
(3) $(b^x)^p = b^{px}$

Interest on C.D.

$A = Pe^{rt}$
P = principal
r = rate
t = time

$r = 6.95\% = .0695$
$P = \$100$, $t = 1$ year

$A \approx \$107.20$
(APY = 7.20%)

Problem

The number of sentenced prisoners in U.S. (Federal and State) per 100,000 residents is given (approximately) by $P(t) = 85.5(1.06)^t = 85.5\,e^{0.058t}$

For what value of t is $P(t) = 300$?

We introduce the logarithm functions.

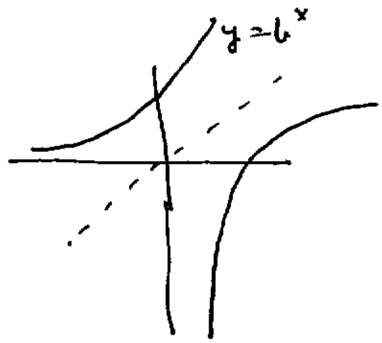
If $y = b^x$ and y is given, what is x?

$10^1 = 10$
$10^2 = 100$
$10^? = 37$

Def: $y = \log_b x$ means $b^y = x$

ie $10^y = 37$ can be written $y = \log_{10} 37$

Graphically



$\boxed{x = b^y \text{ or } y = \log_b x}$

Notation: $\log x$ usually means $\log_{10} x$ [Check this in article, book, etc.]

$\ln x$ " " $\log_e x$

Note: Calculator F223 has a few values for $\ln x$

Problem: $P(t) = 85.5\,e^{0.058t}$

$300 = 85.5\,e^{0.058t}$

$e^{0.058t} = \frac{300}{85.5}$

$0.058t = \ln \frac{300}{85.5}$

$\therefore t = \frac{1}{0.058} \ln \frac{300}{85.5} \approx 21.4$

ie in about 1992

$\boxed{\begin{array}{l} e^a = N \\ \ln N = a \end{array}}$

[U.S. Dept Justice report Nov 3/13 for Dec 31, 1991]

DAY 5

Problem:

The consumption of fuelwood in Ethiopia is given (approximately) by

$$C(t) = 29.2\, e^{0.05t} \qquad (t=0 \text{ is } 1980) \qquad C \text{ is million cubic meters.}$$

when will the consumption be 100 million cubic meters? (assuming same rate of consumption).

ie. For what t is $29.2\, e^{0.05t} = 100$?

Logarithm Function

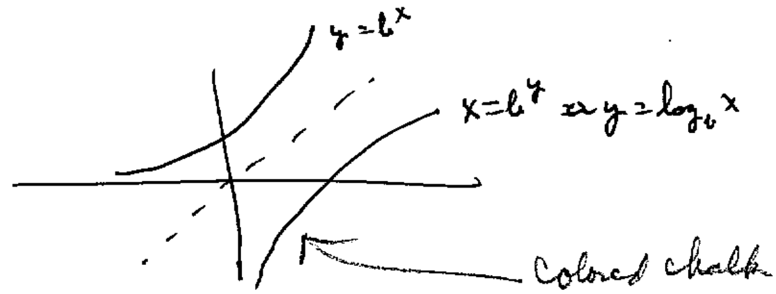
def $y = b^x$ and y is given, what is x?

$10^1 = 10$
$10^2 = 100$
$10^? = 37.$

Def: $y = \log_b x$ means $b^y = x$

ie $10^y = 37$ is written $y = \log_{10} 37.$

Graphically



NOTATION $\log x$ usually mean $\log_{10} x$
$\ln x$ mean $\log_e x$

Calculator has both.
F223 has a few values for $\ln x$

Properties of Logarithm here?

Back to the Problem

$$C(t) = 29.2\, e^{0.05t}$$
$$100 = 29.2\, e^{0.05t}$$
$$e^{0.05t} = \frac{100}{29.2}$$
$$0.05t = \ln \frac{100}{29.2}$$
$$t = \frac{1}{0.05} \ln \frac{100}{29.2} \approx 24.6$$

$\therefore 1980 + 24.6 \approx 2004.6$

ie. a bit past midyear in 2004.

Bureau of Justice Statistics, U.S. Dept. of Justice.

year	number of sentenced prisoners under jurisdiction of State and Federal correctional authorities per 100,000 residents
1970	96
1990	~~297~~
~~2000~~	469

note: plotting data for each year from 1970 seems to indicate an exponential growth.

(a) Estimate r.

$$P = P_0 e^{rt}$$

$t=0\,(1970)$ $\quad 96 = P_0 e^{r\cdot 0} = P_0 \cdot 1 \quad \therefore P_0 = 96$

$$\therefore P = 96 e^{rt}$$

~~$t = 20\,(1990)$~~ $\quad 297 = 96 e^{r \cdot 20}$

$t = 30\,(2000)$

$\therefore e^{20r} = \frac{297}{96}$

$20r = \ln \frac{297}{96}$

$r = \frac{1}{20} \ln \frac{297}{96} \approx 0.056$

$\therefore$ rate is approx 5.6%

$469 = 96 e^{30r}$

$30r = \ln \frac{469}{96}$

$r = \frac{1}{30} \ln \frac{469}{96} \approx 0.053$

$\therefore r \approx 5.3\%$ $\quad$ note $e^{0.053} \approx 1.05$

(b) Now estimate for year ~~1999~~ 1998

$P = 96 e^{.05\cancel{6}t}$

$t = 28$ $\quad P = 96 e^{.056(28)} \approx$ ~~[illegible]~~ 460.5

[~~Preliminary estimate~~ Report for Dept of Justice is ~~[illegible]~~ 461]

Derivative

$$y = e^{u(x)} \qquad \frac{dy}{dx} = e^{u(x)} \cdot \frac{du}{dx}$$

e.g. ① $y = e^{x}$ $\quad \frac{dy}{dx} = e^{x}$

② $y = e^{3x^2}$ $\quad \frac{dy}{dx} = 6x\,e^{3x^2}$

③ $y = x^2 \cdot e^{-x}$ $\quad y' = \frac{dy}{dx} = x^2(-e^{-x}) + 2x e^{-x} = x e^{-x}(2-x)$

$\therefore$ y has stationary points at $x=0$, $x=2$.
(loc. min) (loc. max)

$y'' = e^{-x}(2-4x+x^2)$

④ $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

normal density function with mean μ and standard deviation σ.

$$y' = \frac{dy}{dx} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} (-1)\left(\frac{x-\mu}{\sigma}\right)\cdot\frac{1}{\sigma} = -\frac{1}{\sigma^3\sqrt{2\pi}}(x-\mu)e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$y'' = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\left(1 - \frac{(x-\mu)^2}{\sigma^2}\right)$$

y has max. at $x = \mu$

and inflection pts at $x = \mu \pm \sigma$

Note: $y = e^x$ is positive, increasing, and concave up for all x.

Question: $c(t) = 76.2\,e^{0.03t}$

when is $c = 125$? 130? 135?

Recall: $c(15) = 119.5$
$c(20) = 138.8$

Similarly: $y = e^x$. $\quad x = 2 \Rightarrow y = 7.39$

For what value of x is $y = 2$?

EXAMPLE (Goldstein, Lay, Schneider *Brief Calc.* p327 # 8).

already now? (~~Mexico City is expected to become the most heavily populated city in the world by the end of the century. At the beginning of 1990, 20.2 million people lived in the metropolitan area. At the beginning of 1995, about 23.7 million people lived there. Estimate the exponential growth rate (part is due to immigration).~~

$$P = P_0 e^{rt}$$

$t=0$ (1990) $\quad P = P_0 = 20.2 \qquad \therefore P = 20.2\, e^{rt}$

$t=5$ (1995) $\quad P = 23.7 = 20.2\, e^{5r}$

$$\therefore e^{5r} = \frac{23.7}{20.2} \Rightarrow r = \frac{1}{5} \ln \frac{23.7}{20.2} \approx 0.032$$

$\therefore$ approximately 3.2%

note: $t = 10$ (2000) $\quad P \approx 27.8$ million (low: 30 million projected)

SKIP

Sept 17

Derivative

① $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

$y = \ln x$ means $e^y = x$

$$\therefore e^y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

note: graph of $y = \ln x$ is increasing and concave down for all $x > 0$.

② $y = \ln u(x)$

$$\therefore \frac{dy}{dx} = \frac{1}{u(x)} u'(x)$$

e.g. $y = \ln(x^2+3x-2) \Rightarrow \dfrac{dy}{dx} = \dfrac{2x+3}{x^2+3x-2}$

Properties of Logarithm

Do earlier

① $\log_b uv = \log_b u + \log_b v$

② $\log_b \dfrac{u}{v} = \log_b u - \log_b v$

③ $\log_b u^p = p \log_b u$.

$\log_b uv = m \quad \therefore b^m = uv$

$\log_b u = p \quad \therefore b^p = u$

$\log_b v = q \quad \therefore b^q = v$

$\therefore \underbrace{uv}_{b^m} = b^p b^q = b^{p+q}$.

$\therefore m = p+q$

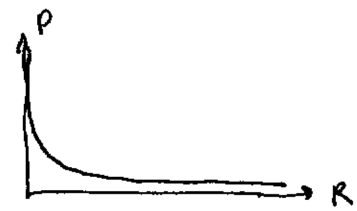
EXAMPLE "ZIPF'S LAW"

- frequency of occurrence of some event (P) as a function of rank (R)
- e.g. English words, population of cities, revenue of companies
- Large events are rare; small ones are common (i.e. few megacities, many small towns)

The rank of the size of an urban area is related to the population of the area ("law of urban concentration")

- few web sites have many visitors, most get few visits.

These data suggest a power curve (not exponential since $R=0$ not defined)



Pareto curve in economics

Suppose $P = cR^k$ (c and k constant).

$$\therefore \log P = \log cR^k$$
$$= \log c + \log R^k$$
$$\log P = \log c + k \log R$$

FORM: $y = b + kx$ i.e. linear in $\log P$ and $\log R$.

Do least squares fit to find k and $\log c$.

i.e. k and c

For the U.S. in ~~[illegible]~~ 2000, there is approximately $P = \cancel{}\; 5.79\, R^{-.74}$

	R	$\hat{P}$ (model)	P (actual)
LA	2	3.311 million	3.633 million
Detroit	10	1.006	.965
St. L	57	.301	.334
Anchorage	63	.258	.258
St. Petersburg FL	70	.238	.230
New Haven, CONN	172	.123	.122
Ann Arbor	203	.108	.110

(OVER)

DAY 6

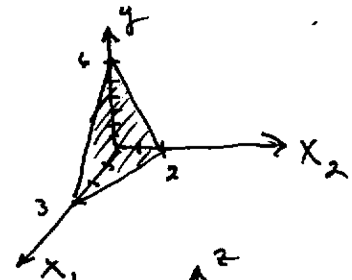
See also: Statistics: A Guide to the Unknown (article by Herbert Simon p240)

PARTIAL DERIVATIVES

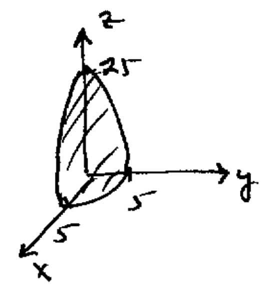
We now consider functions of more than one independent variable.

examples

① $Y = 6 - 2X_1 - 3X_2$ (regression)



② $z = f(x,y) = 25 - x^2 - y^2$

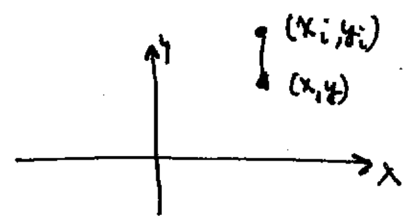


③ $f(x,y) = x^4 + 3xy^2 - 2x^2y + 5y^3 - 4$

④ $f(x,y) = \ln(xy+1)\,\cancel{+(x+y)}\,e^{x^2}$

⑤ $S = (b + mx_1 - y_1)^2 + (b + mx_2 - y_2)^2 + \cdots + (b + mx_n - y_n)^2 = f(m,b)$

x_i, y_i known constants
m, b are unknown parameters
(β_0, β_1)



$y = b + mx$

$b + mx_i - y_i = e_i \qquad S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y - y_i)^2 = \sum_{i=1}^{n} (b + mx_i - y_i)^2$

⑥ $Q = 1.058\, m^{0.136}\, p^{-0.727}\, r^{0.914}\, s^{0.816}$

Data 1929-1941
Richard Stone
[See Goldstein, Lay, Schneider]

skip

m = aggregate real income
p = average retail price of beer
r = average retail price other consumer goods + services
s = strength of beer
Q = quantity of the beer consumed in U.K.

⑦ $R = f(E_R, E_D, I_R, I_D, N)$

see problem #21

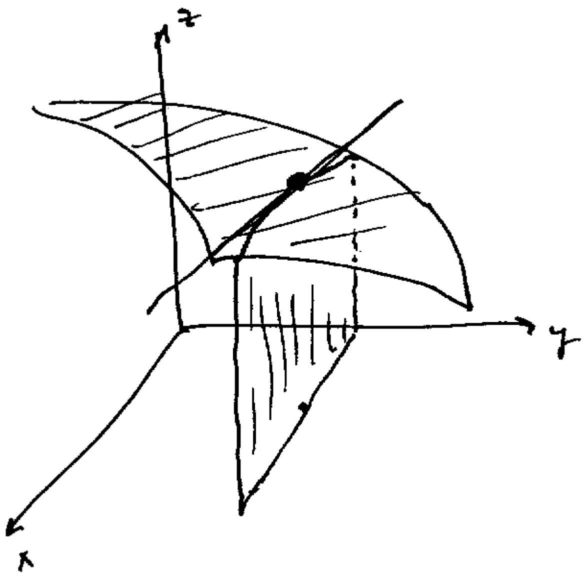
E_R, E_D campaign exp.
I_R, I_D number terms in Congress + 1
N % major vote in 1968

R = Republican % of vote 1974 in a district.

Graphical Interpretation of Derivative

Use umbrella

$z = f(x,y)$



$\dfrac{\partial f}{\partial x} = \dfrac{\partial z}{\partial x} = \left(f_x \right)$ is the partial derivative of f as x changes and y is held fixed

i.e. $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial z}{\partial y} = \left(f_y \right)$ is the partial derivative of f as y changes and x is held fixed

EXAMPLES

① $\dfrac{\partial Y}{\partial X_1} = 2$, $\dfrac{\partial Y}{\partial X_2} = -3$

② $\dfrac{\partial z}{\partial x} = -2x$, $\dfrac{\partial z}{\partial y} = -2y$

③ $\dfrac{\partial f}{\partial x} = 4x^3 + 3y^2 - 4xy$

$\dfrac{\partial f}{\partial y} = 6xy - 2x^2 + 15y^2$

Second Partials

$\dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4y$

$\dfrac{\partial^2 f}{\partial y^2} = 6x + 30y$

$\dfrac{\partial^2 f}{\partial y \partial x} = 6y - 4x = \dfrac{\partial^2 f}{\partial x \partial y}$

④ $\dfrac{\partial f}{\partial x} = \dfrac{\cancel{y}}{xy+1} \cancel{(x+y)} e^{x^2} \cdot 2x + e^{x^2}$; $\dfrac{\partial f}{\partial y} = \dfrac{x}{xy+1} + e^{x^2}$

⑤ $\dfrac{\partial S}{\partial m} = \sum\limits_{i=1}^{n} 2(b + m x_i - y_i) x_i$; $\dfrac{\partial S}{\partial b} = \sum\limits_{i=1}^{4} 2(b + m x_i - y_i) \cdot 1$

⑥ $\dfrac{\partial Q}{\partial m} > 0$; $\dfrac{\partial Q}{\partial p} < 0$

⑦ H.W.

Extrema $z = f(x,y)$

1. Get stationary points, i.e. solve $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$

2. Test (omitted here)

EX: ① none

② $(0,0)$

③ $\begin{cases} \sum_{i=1}^{n} (m x_i^2 + b x_i - x_i y_i) = 0 \\ \sum_{i=1}^{n} (m x_i + b - y_i) = 0 \end{cases}$ i.e. $\begin{cases} m \sum x_i^2 + b \sum x_i = \sum x_i y_i \\ m \sum x_i + b n = \sum y_i \end{cases}$

solve for b and m

(normal equations)

DAY 7 Algebraic Evaluation

98, 99

11:50

Lagrange multipliers

INTEGRATION

Problem 1. Given $y = f(x)$, find $\frac{dy}{dx} = f'(x)$ (differentiation; find derivative) Find change

Problem 2. Given $\frac{dy}{dx} = f'(x)$, find $y = f(x)$ (antidifferentiation; find antiderivative or indefinite integral). Given change.

Note: Problem 2 is, in general, more difficult than Problem 1
However, we can always check the answer if we know how to do problem 1
(i.e. like mult. and division)

Examples

If $\frac{dy}{dx} = x^2 + x - 6$, what is y?

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + \underline{\underline{C}}$$, where C is any constant.
(why called indefinite integral)

Check: $\frac{dy}{dx} = x^2 + x - 6$

Notation: For the above example, we write

$$\int (x^2 + x - 6)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + C$$

TABLES (back of book) p254 der p256 int.

- See the table in Fran 306 for some of these.
- use MATHEMATICA, MAPLE, etc.

Ex:

① $\int (x^3 - 2x + 5)\,dx = \frac{1}{4}x^4 - x^2 + 5x + C$

② $\int (x^{\frac{1}{3}} + x^{-\frac{1}{2}})\,dx = \frac{3}{4}x^{\frac{4}{3}} + 2x^{\frac{1}{2}} + C$

③ $\int e^x\,dx = e^x + C$

④ $\int \frac{1}{x}\,dx = \ln x + C$

or $\int \frac{1}{P}\,dP = \ln P + C$

⑤ ~~[illegible], r a constant~~

$\int r\,dt = rt + C$, r a constant.

Exponential Growth (any population)

— # prisoners

ant Chlorofluorocarbons emitted in U.S. [dense]

Let $P(t)$ be the population at time t and suppose there is a constant rate of growth r. Look at the population at equally spaced times, say Δt (small)

$$\xrightarrow[\quad P(t) \qquad\qquad P(t+\Delta t)\quad]{\Delta t} t$$

We measure the change in P two ways:

① The population change during the small time interval is $P(t+\Delta t) - P(t)$

② Another way to view this change is Population at t × rate × time ["People × rate × time"]

i.e. $\approx P(t)\, r\, \Delta t$ Assume P not constant

These should be "approximately equal" so

$$P(t+\Delta t) - P(t) \approx P(t)\, r\, \Delta t$$

$$\therefore \frac{P(t+\Delta t) - P(t)}{\Delta t} \approx P(t)\, r$$

We improve the model by taking Δt very small, i.e. take the limit as $\Delta t \to 0$

$$\therefore \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \lim_{\Delta t \to 0} P(t) \cdot r$$

$$\therefore \frac{dP}{dt} = P r = r P$$

i.e. the "rate of change of P is proportional to P" [Differential Equation]

$$\frac{dP}{P} = r\, dt \qquad \circledast$$

Differentials F 262-266
$y = f(x)$
$dy = f'(x)\, dx$ so $dP = \frac{dP}{dt} dt$

$$\int \frac{1}{P} dP = \int r\, dt$$

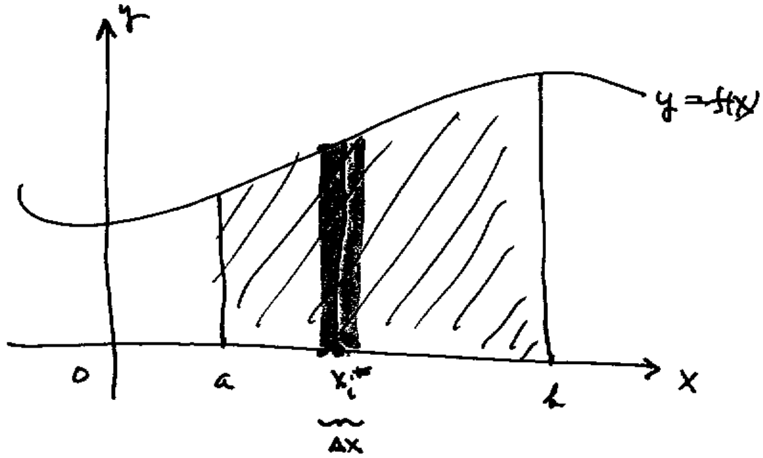
$$\ln P = rt + c$$

$$P = e^{rt+c} = e^{c} \cdot e^{rt} = P_0 e^{rt}$$

check: $\frac{dP}{dt} = P_0 r e^{rt} = rP$

We have solved the differential equation $\circledast$ using integration.

Problem: Find the area A under the curve $y = f(x)$ from $x = a$ to $x = b$



Start with squares? Graph paper

1. approximate by using the area of inscribed rectangles (n of them) (circumscribed)

$$A \approx \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{height}} \underbrace{\Delta x}_{\text{width}} \qquad \left(\sum_{i=1}^{n} f(x_i^{**}) \Delta x\right)$$

2. Take the limit as n gets larger, i.e. Δx gets smaller

$$\lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

Sketches "Start" with squares?

3. Suppose this limit exists, Write $\int_a^b f(x)dx$ for this limit

This is called the definite integral.

$$A = \int_a^b f(x)\, dx$$

$\Sigma \to S \to \int$

Question: How do you actually get this number (limit) if it exists?

<u>The Fundamental Theorem of Calculus</u> a "top 10" theorem.

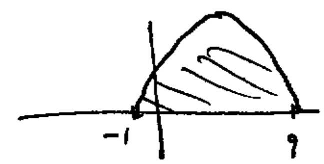
$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b \text{, where } F(x) \text{ is any}$$

antiderivative of $f(x)$, i.e. $F'(x) = f(x)$

Note: There may not always be such an $F(x)$ and then numerical and other techniques are used.

<u>EXAMPLES</u>

① $\displaystyle\int_{-1}^{9} (9+8x-x^2)\,dx$ area under $y = 9+8x-x^2$



$= 9x + 4x^2 - \frac{1}{3}x^3 \Big|_{-1}^{9}$

$= (81 + 324 - 243) - (-9 + 4 + \frac{1}{3})$

$= 167 - \frac{1}{3} = 166\frac{2}{3}$ square units.

② $L(x) = x^{\cancel{}\,2.7}$ $(\cancel{})\,(2000)$

$\therefore\ GI = 1 - 2A = 1 - 2\displaystyle\int_0^1 x^{2.7}\,dx$

$= 1 - 2 \cdot \frac{1}{3.7} x^{3.7}\Big|_0^1$

$= 1 - \frac{2}{3.7} \approx \cancel{0.46} \; 0.46$

other years

2001 0.47
2002 0.46
2003 0.46

③ Find the area under $y = e^{-x}$ from $x = 0$ to $x = 2$

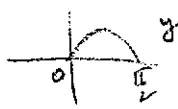
$A = \displaystyle\int_0^2 e^{-x}\,dx = -e^{-x}\Big|_0^2 = -(e^{-2} - e^{0})$

$= 1 - \frac{1}{e^2} \approx .865$

~~④~~ ~~$\int_0^1 e^{-x^2}dx$, e^{-x^2} does not have an antiderivative~~

~~$A \approx 0.7468$~~

(4) $\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

(5) Trig.  $y=\sin(2x)$ $\quad A = \int_0^{\frac{\pi}{2}} \sin(2x)\,dx = -\frac{1}{2}\cos(2x)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}(-1-1) = 1$ sq. unit

~~NOTE: Integration is a summation process. "Area" is sometimes an alias for another quantity.~~

(6) $y=f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$ Normal dist. with mean $\mu=0$ and stand dev $\sigma=1$

$$P(0 \le y \le 1.5) = \int_0^{1.5} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\,dx$$

need numerical approx since $e^{-\frac{1}{2}x^2}$ has no antiderivative.

$$P(0 \le y \le 1.5) \approx 0.4332$$

$$P(-1 \le y \le 1) \approx 0.6827$$

(7) $c(t) = 29.2\, e^{0.05t}$ $\quad t=0$ is 1980

$c(t)$ is rate of fuelwood consumption (mill cu m per year) in ~~Sudan~~ Ethiopia

change

How much consumed from 1980 to 1996

$$T = \int_0^{16} 29.2\, e^{0.05t}\,dt$$

$$= \frac{29.2}{0.05}\, e^{0.05t}\Big|_0^{16} = 584\left(e^{0.8}-1\right)$$

$\approx$ 584 (1.23)

$\approx$ 716 million cubic metres

Note: Integration is a summation process!

"area" is sometimes a proxy for another quantity.
a
alias

(7) Example: Armstrong-Davis p 500
In 1970, 8.6 million tons of waste were recovered for recycling in the U.S.

In 1995, this had increased to 56.2 million tons.
[Statistical abstracts]

The growth appears to be modeled by an exponential

$$W(t) = W_0 e^{rt} \qquad W_0 = 8.6$$
$$r = 0.075$$

$t=0$ is 1970
$t=25$ is 1995

$$\therefore W(t) = 8.6\, e^{0.075t} \qquad \text{[million tons per year]}$$

Determine the total amt of waste recovered for recycling for this period.

$$T = \int_0^{25} 8.6\, e^{0.075t}\, dt$$

$$= \frac{8.6}{0.075} e^{0.075t} \Big|_0^{25}$$

$$= \frac{8.6}{0.075}\left(e^{0.075(25)} - 1 \right)$$

$\therefore T \approx 633$ million tons.