Algebraic Topology; Fundamental Groups

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Some notation

Definition (Path Vs. Loop)

A ***path*** is a continuous map $\alpha : I \to X$. The endpoints are denoted by $x_0 = \alpha(0)$ and $x_1 = \alpha(1)$. If $x_0 = x_1$, then α is said to be a ***loop***. Here, I := [0, 1].

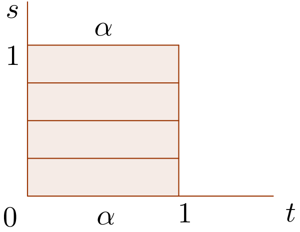
Definition (Homotopy)

Two paths α, $\beta : I \to X$ are said to be ***homotopic***,$\alpha \sim \beta$ if there exists a continuous function $F : I \times I \to X; (t, s) \mapsto x$ with the following properties.

- $F(t, 0) = \alpha(t)$
- $F(t, 1) = \beta(t)$
- $F(0, s) = \alpha(0) = \beta(0)$
- $F(1, s) = \alpha(1) = \beta(1)$

Homotopy is an equivalence relation

Reflexivity is trivial. To show $\alpha \sim \alpha$, just map each intermediate line to α:



Symmetry

Symmetry is also easy. Assume that $\alpha \sim \beta$. Then there is a homotopy $F(t, s)$ that takes α to β. Then define $F'(t, s) = F(t, 1 - s)$. Then F' is a homotopy and $\beta \sim \alpha$.

Transitivity

Assume that F_1 is a homotopy from α to β,and that F_2 is a homotopy from β to γ, for some paths α, β, γ. Then define the following

$$F_3 = \begin{cases} F_1(t, 2s) & s \in [0, \frac{1}{2}] \\ F_2(t, 2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

F_3 is a homotopy from α to γ.

Definition of a Group

A group is a set, X paired with an operation $*$, with the following properties:

1. Closure: $x * y \in X$ for all $x, y \in X$.
2. There is an identity element $e \in X$, such that for any element $x \in X$, $e * x = x * e = x$
3. For every $x \in X$, there is an element $x^{-1} \in X$, such that $x * x^{-1} = e$ for all $x \in X$.
4. Associativity: $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$.

Composition of Paths

Let α and β be two paths in some space X. If $\alpha(1) = \beta(0)$, then define the composition of α and β

$$\alpha * \beta = \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \beta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Composition of loops (with the same base point) is convenient, because for any two loops α, β, $\alpha * \beta$ and $\beta * \alpha$ are both defined.

Identity and Inverse Loop

Definition (Identity Loop)

The ***identity loop*** α_e around a base point x_0 is defined as the constant map $\alpha_e(t) = \alpha_e(0) = x_0$.

Definition (Inverse Loop)

If $\alpha(t)$ is a loop, then the ***inverse loop*** of $\alpha(t)$ is $\alpha^{-1} := \alpha(1-t)$.

α^{-1} is the inverse of α under composition, because $\alpha * \alpha^{-1}$ is *homotopic* to α_e.

Associativity of Loop Composition

As noted previously, for path composition, $\alpha(t) * \beta(t)$, Where $\alpha(t)$ ends $\beta(t)$ has to begin. Loops begin and end at the same element, so the composition of loops is associative. Associativity has to be proven by showing that for three loops, α, β, γ, there is a homotopy from $\alpha * (\beta * \gamma)$ to $(\alpha * \beta) * \gamma$

Forming a Group with Loops

Let $[\alpha]$ denote the equivalence class of a loop α. Let $\Lambda(X, x_0)$ be the set of equivalence classes of a space X, based at a point x_0. From the prior slides, we can conclude that Λ is a group under the operation of composition. (Closure is fairly obvious)
One of the basic ideas of algebraic topology is that we can classify shapes and surfaces by "assigning" algebraic groups to them.

Example-S^2 and $\mathbb{R}^1$

- These both have the same fundamental group...

Example-S^2 and $\mathbb{R}^1$

- These both have the same fundamental group...
- ...but are not homeomorphic.

Example - Torus

Claim 1: $\pi(T, x_0) = \mathbb{Z} \times \mathbb{Z}$
Let's show this on the board, I couldn't load the packages to generate the torus.

Topological Invariant

Definition (Topological Invariant)

A ***topological invariant*** is a property or structure of a topological space that is preserved under a homeomorphism.

Topological invariants of a topological space X are shared among any topological space homeomorphic to X.

To prove that the fundamental group is a topological invariant, we need to show that if a topological space X is homeomorphic to a topological space Y, then the fundamental group of X, denoted $\pi(X)$ is isomorphic to the $\pi(Y)$, the fundamental group of Y.

$$X \approx Y \Longrightarrow \pi(X) \approx \pi(Y)$$

Notice that the inverse is not true

$$\pi(X) \approx \pi(Y) \not\Rightarrow X \approx Y$$

Induced Homomorphism

Definition (Induced Homomorphism)

Given a semi-continuous map $f : X \longrightarrow Y$, the ***induced homomorphism*** on $\pi(X * Y)$ is an equivalence class of endomorphisms on the sub-manifold formed by all half-disjoint subsets of $X * Y$.