Understanding Analysis excerises and solutions

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1.2.6. Given a function f and a subset of A of its domain, left $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$, and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$ in this case?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets, $A, B, \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

solution: **(a)** I have $f(A) = f([0, 2]) = [0, 4]$ and $f(B) = f([1, 4]) = [1, 16]$. The intersection of the images of A and B is $f(A) \cap f(B) = [0, 4] \cap [1, 4] = [1, 4]$. The image of the intersection of $A \cap B$ is $f(A \cap B) = f([1, 2]) = [1, 4]$. In this case, the two are equal.

Now looking at unions, we see that $f(A \cup B) = f([0, 4]) = [0, 16]$. Further, $f(A) \cup f(B) = [0, 4] \cup [1, 16] = [0, 16]$.

(b) Let $A = [-1, 0]$ and let $B = [0, 1]$. Then $A \cap B = [0] = \{0\}$ and f(A) = f(B) = [0,1]. Then:

$$f(A) \cap f(B) = [0, 1] \neq \{0\} = f(\{0\}) = f(A \cap B)$$

(c) Show that $g(A \cap B) \subseteq g(A) \cap G(B)$.

Proof. Let $z \in g(A \cap B) = \{f(x) | x \in (A \cap B)\}$. This means $z = f(x)$, for some $x \in A \cap B$. Thus:

1. $x \in A \cap B \;\Rightarrow\; x \in A \;\Rightarrow\; f(x) \in f(A)$
2. $x \in A \cap B \;\Rightarrow\; x \in B \;\Rightarrow\; f(x) \in f(B)$

From 1 and 2 above, it follows that $f(x) \in f(A) \cap f(B)$ as desired. □

(d) Claim: $g(A) \cup g(B) = g(A \cup B)$

Proof. First the forward inclusion. Let $z \in g(A) \cup g(B)$. Then $z \in g(A)$ or $z \in g(B)$ which implies that $z = g(x)$, for some $x \in A$ or $x \in B$, that is, for some $x \in A \cup B$. Thus $z \in \{g(x) | x \in A \cup B\} = f(A \cup B)$.
Now for the backwards inclusion. Let $z \in g(A \cup B)$. Then $z = g(x)$ for some $x \in A \cup B$, implying $z = g(x) \in g(A)$ or $z = g(x) \in g(B)$. That is, $z \in \{g(x) | x \in A\}$ or $z \in \{g(x) | x \in B\}$, implying $z \in g(A) \cup g(B)$. □

1.2.7. Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, left $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$, $A = [0, 4]$, and $B = [-1, 1]$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ in this case?

(b) The attributes demonstrated in (a) are general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ for all sets $A, B \subseteq \mathbb{R}$. Do the same for unions.

solution **(a)** First I write the preimages of A and B. $f^{-1}(A) = f^{-1}([0, 4]) = [-2, 2]$ and $f^{-1}(B) = f^{-1}([-1, 1]) = [-1, 1]$. I also have $f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1] = [-2, 2] \cap [-1, 1]$, the intersection of the preimages. Next, $f^{-1}(A) \cup f^{-1}(B) = [-2, 2] = f^{-1}([-1, 4]) = f^{-1}(A \cup B)$.

(b) Show that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$

Proof. Let $z \in g^{-1}(A \cap B) = \{x \mid g(x) \in A \cap B\}$. This means $g(z) \in A$ and $g(z) \in B$, or $z \in g^{-1}(A) \cap g^{-1}(B)$. Now let $z \in g^{-1}(A) \cap g^{-1}(B)$. Then $g(z) \in A$ and $g(z) \in B$, so $z \in \{x : g(x) \in A \text{ and } g(x) \in B\} = g^{-1}(A \cap B)$. □

Show that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$.

Proof. First, the forward inclusion. Let $z \in g^{-1}(A \cup B) = \{x \in \mathbb{R} : g(x) \in A \cup B\}$. Then $g(z) \in A \cup B$, or in other words, $g(z) \in A$ or $g(z) \in B$. This implies that $z \in g^{-1}(A)$ or $z \in g^{-1}(B) \equiv z \in g^{-1}(A) \cup g^{-1}(B)$.

Now let $z \in g^{-1}(A) \cup g^{-1}(B)$. Then $g(z) \in A$ or $g(z) \in B$, which is equivalent to writing $g(z) \in A \cup B$. This implies, however, that $z \in g^{-1}(A \cup B)$, and the proof is complete. □

1.3.2.

(a) Write a formal definition in the stle of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.7 for greatest lower bounds.

solution **(a) Definition 1.32** (for inima) A real number t is the greatest lowest bound for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) t is a lower bound for A.

(ii) if c is any lower bound for A, then $c \le t \implies (c > t \Rightarrow$ c is not a lower bound[1]).

(b) *Lemma 1.3.7 (For infima)* Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. Then, $t = \inf(A)$ iff $\forall \epsilon > 0, \exists a \in A$, such that $t + \epsilon > a$.

Proof. First assume that $t = \inf(A)$. Now let $\epsilon > 0$ be a real number. By property(ii) in the definition from part (a) above, $t + \epsilon$ is not a lower bound, since $t + \epsilon > t$. Since $t + \epsilon$ is not a lower bound, there must be an $a \in A$, such that $a < t + \epsilon$. This proves the forward direction.

Now assume that for all $\epsilon > 0$, that there is an $a \in A$ such that $t + \epsilon > a$. Now let $c > t$. I choose $\epsilon := c - t$ Then there is an $a \in A$, such that $t + (c - t) = c > a$. This demonstrates property (ii) for being a lower bound –that c is not a lower bound for A. The first property (that t is a lower bound) is satisfied by assumption, so $t = \inf A$. □

[1] I put this here for reference in part (b)

1.3.3.

(a) Let A be bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Show the $\sup B = \inf A$.

Proof. Let $s = \sup(B)$. How do we know s exists? Because B is bounded above (take any $a \in A$, since $b \leq a, \forall a \in A$), there is a supremum. s has the following two properties by definition of suprermum:

(i) $b \leq s$, $\forall b \in B$ (s is an upperbound for B)

(ii) c is an upperbound for $B \Rightarrow c \geq s$

In order to show that s is the infimum of A, I need to show that is has the following two properties:

(a) $s \leq a, \forall a \in A$($s$ is a lower bound for A)

(b) c is a lower bound for $A \Rightarrow c \leq s$

First I will show that s is a lower bound for A. Since each $b \in B$ is a lower bound for A, i.e. $b \leq a, \forall a \in A$, each $a \in A$ is an upperbound for B. Therefore, by property (ii) above, $s \leq a$, for all $a \in A$, as desired.

Now I will prove that the second property holds. Let c be a lower bound for A. I need to show $c \leq s$. Since c is a lower bound for A, $c \in B$ by definition. But by property (i) of supremum, $b \leq s \Rightarrow c \leq s$.

This proves that $s = \sup(B) = \inf(A)$ □

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

We do not have to assert that the infimum exists, precisely because of the process in (a). Let $A \subseteq \mathbb{R}$, where A is bounded below. The set of lower bounds for A is well defined, and is clearly bounded above, meaning the set has a supremum. I showed that this supremum must have the property of being the infimum of A. Thus, if supremums exist for sets bounded above, then infimums must exist for sets with lower bounds.

(c) Propose another way to use the Axiom of Completeness to prove that sets bounded below have greatest lower bounds.

Let $(a, b) \subset \mathbb{R}$, with $a \leq b$. Simililar to part (a), it can be shown that $\sup\Big((-\infty, a]\Big) = \inf\Big((a, b)\Big)$, implying the existence of greatest lower bounds for sets bounded below.

1.3.6. Compute, without proofs, the suprema and infima of the following sets:

(a) $\{n \in \mathbb{N} : n^2 < 10\}$

(b) $\{\frac{n}{m+n} : m, n \in \mathbb{N}\}$

(c) $\{\frac{n}{2n+1} : n \in \mathbb{N}\}$

(d) $\{\frac{n}{m} : m, n \in \mathbb{N} \text{ with } m + n \leq 10\}$

solution

(a) $\inf = 1$, $\sup = 3$

(b) $\inf = 0$, $\sup = 1$

(c) $\inf = \frac{1}{3}$, $\sup = \frac{1}{2}$

(d) $\inf = \frac{1}{9}$, $\sup = 9$

2.2.1 (b) Show that $\lim(a_n) = \lim \dfrac{3n+1}{2n+5} = \dfrac{3}{2}$

Proof sketch: First we write

$$|a_n - L| = \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{6n+2-6n-15}{2(2n+5)}\right| = \frac{13}{4n+10}$$

If we want $|a_n - L| = \dfrac{13}{4n+10} < \epsilon$, then solving the inequality for n will provide a suitable choice for N.

$$\begin{aligned} \frac{13}{4n+10} &< \epsilon \\ 13 &< 4n\epsilon + 10\epsilon \\ \frac{13-10\epsilon}{4\epsilon} &< n \end{aligned}$$

So for $n \geq N = \dfrac{13-10\epsilon}{4\epsilon}$, we are ready.

Proof. Let $\epsilon > 0$ be given. Then for all natural numbers n greater than $\dfrac{13-10\epsilon}{4\epsilon}$:

$$|a_n - L| = \left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \frac{13}{4n+10} < \frac{13}{4(\frac{13-10\epsilon}{4\epsilon})+10} = \frac{13}{\frac{13-10\epsilon+10\epsilon}{\epsilon}} = \epsilon$$

Therefore the limit is $\dfrac{3}{2}$ □

2.2.2 A sequence x_n vercongess to x if there exists an $\epsilon > 0$ such that for all N in $\mathbb{N}$, $n \geq N$ implies $|x_n - x| < \epsilon$. The definition states, "for all N in $\mathbb{N}, n \geq N$ implies $\ldots$". Since $1 \in \mathbb{N}$, the definition could be rewritten to say that a sequence x_n verconges to x if for all $n \in \mathbb{N}$, $|x_n - x| < \epsilon$, for some positive ϵ.

Theorem (vercongence): Suppose (a_n) is bounded. Then (a_n) verconges to all real numbers.

Proof. Let (a_n) be bound above, and let $M > sup(a_n)$ be some upper bound of (a_n). Now let a be some real number. To show that (a_n) verconges to a, choose $\epsilon = M + |a|$. Then for all $n \in \mathbb{N}$,

$$|a_n - a| \leq |a_n| + |-a| = |a_n| + |a| < M + |a| = \epsilon$$

Therefore (a_n) verconges to a, and so for all real numbers. □

Any bounded sequence verconges. Every convergent sequence is vercongent, but not every vercongent sequence is convergent. For example, the sequence $1, 0, 1, 0, 1, 0, 1, 0, \ldots$ verconges to all real numbers, but does not converge.

2.2.4 The sequence $1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots$ can be rewritten as:

$$a_n = \begin{cases} 1 & \text{if } n = \dfrac{k(k+1)}{2} \quad k \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$$

This series cannot converge to zero. Given the jth term of the sequence, if we define $l := \dfrac{j(j+1)}{2}$, then $a_j \leq a_l = 1$. For ϵ greater than one, there is no problem in choosing an N, since each term is either 1 or 0, so that $|a_n| < \epsilon$ for all n. For $\epsilon \leq 1$, however, no such N can be found, precisely because of the fact mentioned above– that the sequence will continue to visit 1 indefinitely.

2.2.7 A sequence a_n converges to infinity, if for each $M \in \mathbb{N}$ (or $\mathbb{R}$), there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > M$. So to prove that $\sqrt{x}$ converges to infinity, the obvious choice given some number M is to choose N to be greater than M^2, implying that $\sqrt{n} > M$, for all $n \geq N$.

Our intuition would tell us that the given sequence diverges to infinity. However, the definition above says this is not the case. There is no way to choose an N so that the sequence is always greater than some given number, since it will constantly return to zero.

2.3.2 Let $x_n \geq 0$ for all $n \in \mathbb{N}$. Show that if $x_n \longrightarrow 0$ then $\sqrt{x_n} \longrightarrow \sqrt{x}$. Since x_n is non-negative, we can make $|x_n - 0| = x_n < \epsilon$, for any given positive number ϵ.

Proof. Let ϵ be a given positive number. We want to show that n greater than some N, that $|\sqrt{x_n}| < \epsilon$. If we consider the quantity ϵ^2, then for some $N \in \mathbb{N}$, and $n \geq N$, $x_n < \epsilon^2$. Taking the square root of both sides, we get $\sqrt{x_n} < \epsilon$, as desired. The N that we need to choose is the same one we chose to make $x_n < \epsilon^2$. □

If $x_n \longrightarrow x$ then $\sqrt{x_n} \longrightarrow \sqrt{x}$.

Proof. Let $\epsilon > 0$ be given. We want to show that for n greater than some N, $|\sqrt{x_n} - \sqrt{x}| < \epsilon$. Since we are given that we can make $|x_n - x|$ as small as we like, the idea is to get $|\sqrt{x_n} - \sqrt{x}|$ to look something like $|x_n - x|$. To do this, multiply the top and bottom by its conjugate:

$$|\sqrt{x_n} - \sqrt{x}|\frac{|\sqrt{x_n} + \sqrt{x}|}{|\sqrt{x_n} + \sqrt{x}|} = \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|}$$

The expression on the right can be compared to $\dfrac{|x_n - x|}{|\sqrt{x_n}|}$ or $\dfrac{|x_n - x|}{|\sqrt{x}|}$, since the original will be smaller. The latter is more "friendly", since $\sqrt{x}$ is just a number. Then I define $\epsilon_0 = \sqrt{x} \cdot \epsilon$. Then for some $N \in \mathbb{N}$, $|x_n - x| < \epsilon_0$. Therefore, we can deduce the following:

$$\frac{|x_n + x|}{|\sqrt{x_n} + \sqrt{x}|} \leq \frac{|x_n - x|}{|\sqrt{x}|} < \frac{\epsilon_0}{\sqrt{x}} = \epsilon\frac{\sqrt{x}}{\sqrt{x}} = \epsilon$$

□

2.3.4 Show that limits must be unique.

Proof. Assume that $L_1 \neq L_2$. Then by definition of limit, the interval $(L_1 - \dfrac{|L_1 - L_2|}{2}, L_1 + \dfrac{|L_1 - L_2|}{2})$ and the interval $(L_2 - \dfrac{|L_1 - L_2|}{2}, L_2 + \dfrac{|L_1 - L_2|}{2})$ would both contain all but a finite number of of the memebers of the sequence. This would imply that both intervals –which are disjoint– would contain an infinite number of elements of x_n. This is a contradiction, so $L_1 = L_2$. □

2.3.8

(a) Let $x_n = 0, 1, 2, 3, 4, 5, \ldots$ and let $y_n = 0, -1, -2, -3, -4, \ldots$. Both (of these sequences diverge, but their sum is the zero sequence, which is convergent.

(b) This is impossible. Suppose $x_n \to x$ and $(x_n + y_n) \to c$. Then by theorem 2.3.3, $\lim \big[(x_n + y_n) - x_n\big] = c - a = \lim y_n$. This is a problem, since y_n cannot converge to $(c - a)$, since $(c - a)$ is finite.

(c) Let $b_n = \dfrac{1}{n^k}$, where $k \geq 1$. Clearly this converges, but $\lim \dfrac{1}{b_n}$ is divergent.

(d) If b_n is convergent, then b_n is bounded. Since a_n is unbounded, for each $M \geq \sup(b_n)$, $a_n \geq M$ for some N. Therefore the difference between a_n and b_n will become arbitrarily high, so $a_n - b_n$ cannot be bounded.

(e) Let $a_n = \dfrac{1}{n^2}$ and let $b_n = n$. Then (a_n) is convergent, (b_n) is divergent, and $(a_n \cdot b_n) = \dfrac{1}{n}$ is convergent.

Extension of Cesaro's theorem: Let (a_n) and (b_n) be sequences such that $b_n > 0$ for all n, the seiries $\sum_{n=1}^{\infty} b_n$ diverges to ∞ and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$. Prove :

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = L$$

Proof. First we start by noting the following:

$$\begin{aligned}
\left| \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} - L \right| &= \left| \frac{\sum_{k=1}^n a_k - L(\sum_{k=1}^n b_k)}{\sum_{k=1}^n b_k} \right| \\
&= \left| \frac{\sum_{k=1}^n [a_k - L \cdot b_k]}{\sum_{k=1}^n b_k)} \right| \\
&\leq \frac{1}{\sum_{k=1}^n b_k} \sum_{k=1}^{n} |a_k - L \cdot b_k| \\
&= \frac{1}{\sum_{k=1}^n b_k} \sum_{k=1}^{t} |a_k - L \cdot b_k| + \frac{1}{\sum_{k=1}^n b_k} \sum_{k=t+1}^{n} |a_k - L \cdot b_k|
\end{aligned}$$

Here, we break up the second to last equation by partitioning it into two halves. By hypothesis, we know that we can make $\left| \dfrac{a_n}{b_n} - L \right| = \left| \dfrac{a_n - L \cdot b_n}{b_n} \right|$ to be as small as we like by choosing n large enough. Let's choose t so that $\left| \dfrac{a_n - L \cdot b_n}{b_n} \right| < \dfrac{\epsilon}{2}$ for $n \geq t$. We also know that $a_k - L \cdot b_k$ is finite, since L is finite, and we are only taking a finite number of terms from each sequence. Therefore, since we are dividing by a diverging series, we will be able to choose n large enough that $\dfrac{1}{\sum_{k=1}^n b_k} \sum_{k=1}^{t} |a_k - L \cdot b_k| < \dfrac{\epsilon}{2}$. We can now conclude the following:

$$\left| \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} - L \right| \leq \frac{1}{\sum_{k=1}^n b_k} \sum_{k=1}^{t} |a_k - L \cdot b_k| + \frac{1}{\sum_{k=1}^n b_k} \sum_{k=t+1}^{n} |a_k - L \cdot b_k| < \frac{\epsilon}{2} + \frac{n-t}{n} \cdot \frac{\epsilon}{2} < \epsilon$$

□

2.4.2

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) of this excercise to explicitly compute $\lim x_n$.

Solution

(a) My first claim is that x_n is decreasing, and I will do this be induction:

Proof. Since $x_2 = 1$, and $1 < 3 = x_1$, the claim holds for $n = 1$. Now suppose that $x_k < x_{k+1}$. We want to show that $x_{k+1} < x_{k+2}$. Then we observe the following:

$$\begin{aligned} x_k &< x_{k+1} && \text{(By the induction hypothesis)} \\ 4 - x_k &< 4x_{k+1} \\ \frac{1}{4 - x_k} &> \frac{1}{4 - x_{k+1}} \end{aligned}$$

The terms on the left and right are, by definition, $x_k{+}1$ and x_{k+2} respectively, which gives $x_{k+1} > x_{k+2}$. Therefore we have a decreasing function.

It is fairly obvious that each x_n will be greater than zero, since each x_n is less than 3, implying each x_n is a positive rational number. By the monotone convergence theorem, this sequence converges. □

(b) $\lim x_{n+1}$ exists because removing a finite amount from $\lim x_n$ will have no effect on the convergence.

(c) if $L = \lim x_n = \lim x_{n+1}$, then

$$\begin{aligned} L &= \frac{1}{4 - L} \\ -L^2 + 4L - 1 &= 0 \\ L^2 - 4L + 1 &= 0 \end{aligned}$$

So L is $2 \pm \sqrt{3}$. We can't have $L = 2 + \sqrt{3}$, because each x_n is less the the 3, the first term, leaving $L = 2 - \sqrt{3}$.

2.4.6 Let a_n be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The limit superior of a_n is defined by

$$\limsup a_n = \lim y_n$$

Provide a reasonable definition for $\liminf a_n$

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ iff $\lim a_n$ exists.

Solution

(a) *Proof.* First, since a_n is bounded, there is an M such that $|a_n| < M$ for all n. It follows that y_n must also be bounded, since each y_n is the supremum of a collection of elements from a_n. Next we need to show that y_n is monotone. Our claim is that y_n is non-increasing. Consider the sets $\{a_k : k \geq n\}$ and $\{a_k : k \geq n + 1\}$. The second set is clearly a subset of the first, since all we've done is remove an element of the first. For convienence, let us give a name to these sets:

$$I_n = \{a_k : k \geq n\}$$

We want to show that y_n is non-increasing, which amounts to showing that $\sup I_n \geq \sup I_{n+1}$. This is easy since $I_{n+1} \subset I_n$ implies that each element in I_{n+1} must be less than or equal to each element in I_n. Therefore, $\sup I_{n+1}$ cannot be greater than $\sup I_n$. Since y_n is monotone and bounded, it must converge. □

(b) We define similar to part (a):

$$x_n = \inf\{a_k : k \geq n\}$$

From this, we can define the limit infinum of a_n to be $\lim x_n$. Showing that x_n converges is identical to the proof of y_n converging, only it follows that x_n is non-decreasing instead of non-increasing.

(c) Since each $x_n \leq y_n$, it follows that $\lim x_n \leq \lim y_n$ (from Theorem 2.3.4 (ii))

Consider the sequence $s_n = -1^n$. Each $y_n = 1$ and each $x_n = -1$, so the two will never have the same value.

2.5.3

(a) Define

$$a_n = \begin{cases} \dfrac{n}{n+1} & : \text{n is even} \\ \dfrac{1}{n} & : \text{else} \end{cases}$$

(b) This is impossible. Every subsequence of a monotone sequence must also be monotone and unbounded. For the subsequence to converge, it would have to be the case that it was bounded.

(c) The sequence $1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ will converge to each unit fraction.

(d) The sequence $1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, \ldots$ is unbounded but contains the zero sequence.

(e) The Bolzano Weierstrass Theorem states that every bounded sequence contains a convergent subsequence. The bounded subsequence would have to contain a converging subsequence as a result, so this is impossible.

2.5.4 Assume a_n is bounded and has the property that every convergent subsequence of a_n converges to the same limit, call it L. Prove that a_n must converge to L.

Solution

Proof. We will do proof by contradiction. Assume that a_n converges to $Q \neq L$. Then all but a finite number of elements of a_n lie in $(Q - \epsilon, Q + \epsilon)$. Now let a_{n_k} be a subsequence. Then all but a finite number of elements from a_{n_k} are contained in $(L - \epsilon, L + \epsilon)$. Now choose ϵ to be less than half the distance between Q and L. Then all but a finite number of every convergence subsequence would contained in an interval disjoint from $(Q - \epsilon, Q + \epsilon)$. But this implies that there are an infinite number of elements in a_n that are not in $(L - \epsilon, L + \epsilon)$, a contradiction! So a_n converges to L. $\square$

Monotone problem Prove that every sequence has a monotone subsequence.

Proof. Let a_n be a sequence. Using your hint, we define a peak point to be some a_i, so that all subsquent terms are smaller than (or equal to) a_i. If there are inifinitely many, than it is easy. Define a_{n_k} to be the set of peak points. It is clear that each member is smaller than or equal to all of the prior members, so is monotone.

Now suppose that there are only a finite number of peak points. Consider the last of these peaks, let's call it a_j. Then a_{j+1} is greater than or equal to a_j. Since a_{j+1} is not a peak point, we can choose another element, $a_{j+c_1} \geq a_{j+1}$, for some c_1. Since this is not a peak point, we can choose another element $a_{j+c_2} \geq a_{j+c_1}$. We can continue this process indefinitely, providing a non-decreasing subsequence. $\square$

Suppose we are given a bounded sequence, call it b_n. The monotone convergence theorem states that every monotonic bounded sequence converges. From part (a), it follows that b_n has a monotonic subsequence, and we know that it must be bounded (since b_n is bounded), so the subsequence must converge.

2.7.9 Given a series $\sum_{n=1}^{\infty} a_n$ with $(a_n) \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

then the series converges absolutely.

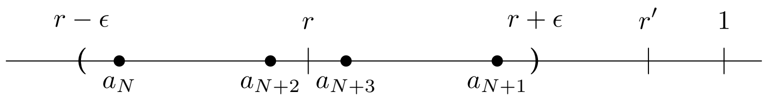
(a) Let r' satisfy $r < r' < 1$. (why must such an r' exist?) Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ necessarily converge?

(c) Now show that $\sum |a_n|$ converges.

solution

(a) Suppose $\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$. Now let r' be greater than r but less than 1. Such a number exists because the reals are dense. Further, choose an ϵ such that $r + \epsilon < r'$. Since $\lim \left| \frac{a_{n+1}}{a_n} \right| = r$, we can choose an N, such that for $n \geq N$, $\left| \frac{a_{n+1}}{a_n} \right| \in (r - \epsilon, r + \epsilon)$.



In other words, for $n \geq N$, $\left| \frac{a_{n+1}}{a_n} \right| < r + \epsilon < r'$. Multiplying through by $|a_n|$ gives

$$|a_{n+1}| < |a_n| r'$$

(b) The sum $|a_N| \sum (r')^n$ converges simply for the fact that it is a geometric series with r' is less than 1 and $|a_N|$ is just a fixed number.

(c) In part (a) we showed that $|a_{n+1}| < |a_n| r'$ for all n greater than some N. Thus note the following observations:

$$\begin{aligned}
|a_{N+1}| &< |a_N| \cdot r' \\
|a_{N+2}| &< |a_{n+1}| r' < \big(|a_N| \cdot r'\big) r' \\
|a_{N+3}| &< |a_{n+2}| r' < \big(|a_{N+1}| \cdot (r')^2\big) r'
\end{aligned}$$

We can write this pattern explicitly as $|a_{N+k}| < |a_N| (r')^k$. Using part (a) and (b), we can conclude that for some N, the following holds:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| < \sum_{n=1}^{N-1} |a_n| + |a_N| \sum_{k=1}^{\infty} (r')^k$$

$\sum_{n=1}^{N-1} |a_n|$ is finite, and the it was shown in (b) that the right most sum is convegent, so the original sum must be convergent.

Limit comparison test Let (a_n) and (b_n) be sequences of positive terms with $\lim_{n\to\infty}\frac{a_n}{b_n} = L > 0$. Then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.

Proof. The fact that $\lim_{n\to\infty}\frac{a_n}{b_n} = L > 0$ means that for every positive ϵ, $\left|\frac{a_n}{b_n} - L\right| < \epsilon$ for n larger than some N, or written differently, $\left|\frac{a_n}{b_n}\right| \in (L-\epsilon, L+\epsilon)$. This implies that for our choice of $n \geq N$, $\frac{a_n}{b_n} < L+\epsilon \Longrightarrow a_n < b_n(L+\epsilon)$. It also implies that $\frac{a_n}{b_n} > L - \epsilon \Longrightarrow b_n < \frac{1}{L-\epsilon}a_n$ for the same choice of $n \geq N$. The following implications can be made:[2]

$$\left[a_n < b_n(L+\epsilon), n \geq N\right] \quad \Longrightarrow \quad \left(\sum b_n(L+\epsilon) \text{ converges} \Rightarrow \sum a_n \text{ converges}\right)$$

$$\left[b_n < \frac{1}{L-\epsilon}a_n, n \geq N\right] \quad \Longrightarrow \quad \left(\sum \frac{1}{L-\epsilon}a_n \text{ converges} \Rightarrow \sum b_n \text{ converges}\right)$$

$$\sum b_n(L+\epsilon)\text{converges} \quad \Longleftrightarrow \quad \sum b_n \text{ converges}$$

$$\sum a_n \frac{1}{(L-\epsilon)}\text{converges} \quad \Longleftrightarrow \quad \sum a_n \text{ converges}$$

Combining these, we get the following two implications:

$$\sum b_n \text{ converges} \Longleftrightarrow \sum b_n(L+\epsilon) \text{ converges} \Longrightarrow \sum a_n \text{ converges}$$

$$\sum a_n \text{ converges} \Longleftrightarrow \sum \frac{1}{L-\epsilon}a_n \text{ converges} \Longrightarrow \sum b_n \text{ converges}$$

Looking at the left and right most ends, we have $\sum b_n$ converges implies $\sum a_n$ converges, and $\sum a_n$ converges implies $\sum b_n$ converges. That both of these hold is equivalent to the if and only if we were seeking. □

2.7.11 Find examples of two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ that both diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples in which (a_n) and (b_n) are positive and decreasing.

solution Though it would be interesting and entertaining to come up with some exotic examples of such sequences, for the sake of homework I chose the simplest path – find a way to make $\sum \min\{a_n, b_n\} = \frac{1}{n^2}$. The idea is to have the sequences "bounce" back and forth between $\frac{1}{n^2}$ and fixed terms. While one sequence has terms of $\frac{1}{n^2}$, the other stays fixed for enough terms that add up to a number (1 is an obvious choice). To be more explicit, if $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$, then set $b_n = \frac{1}{n^2}$ for $n \leq 4$. Then the sum of the first four terms of (a_n) is 1. Now let $b_n = \frac{1}{9}$ for the next 9 terms, so $\sum_{5}^{13} b_n = 1$. For this duration let $(a_n) = \frac{1}{n^2}$. Then repeat the process so the next set of fixed terms of (a_n) add to 1 again, while $b_n = \frac{1}{n^2}$. Both of the series $\sum a_n$ and $\sum b_n$ will diverge, because they will add 1 infinitely many times, but $\sum \min\{a_n, b_n\} = \sum \frac{1}{n^2}$ which converges.

[2] It is probably worth noting that dividing by $L - \epsilon$ is valid since we are free to choose $\epsilon < L$, so $L - \epsilon \neq 0$.

2.7.12 Let (x_n) and (y_n) be sequences, and let $s_n = \sum_{k=1}^{n} x_k$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m+1}^{n} x_j y_j = s_n y_n - s_m y_{m+1} + \sum_{j=m+1}^{n-1} s_j (y_j - y_{j+1})$$

.

solution We start by rewriting the $\sum x_j y_j$ as follows:

$$\begin{aligned}
\sum_{j=m+1}^{n} &= x_{m+1}y_{m+1} + x_{m+2}y_{m+2} + x_{m+3}y_{m+3} + \cdots + x_{n-2}y_{n-2} + x_{n-1}y_{n-1} + x_n y_n \\
&= (s_{m+1} - s_m)y_{m+1} + (s_{m+2} - s_{m+1})y_{m+2} + (s_{m+3} - s_{m+2})y_{m+3} + \cdots + (s_{n-1} - s_{n-2})y_{n-1} + (s_n - s_{n-1})y_n \\
&= s_{m+1}y_{m+1} - \underline{s_m y_{m+1}} + s_{m+2}y_{m+2} - s_{m+1}y_{m+2} + s_{m+3}y_{m+3} + \cdots + s_{n-1}y_{n-1} - s_{n-2}y_{n-1} + \underline{s_n y_n} - s_{n-1}y_n \\
&= \left[s_n y_n - s_m y_{m+1}\right] + s_{m+1}y_{m+1} + s_{m+2}y_{m+2} - s_{m+1}y_{m+2} + \cdots + s_{n-1}y_{n-1} - s_{n-2}y_{n-1} - s_{n-1}y_n \\
&= \left[s_n y_n - s_m y_{m+1}\right] + s_{m+1}(y_{m+1} - y_{m+2}) + s_{m+2}(y_{m+2} - y_{m+3}) + \cdots + s_{n-1}(y_{n-1} - y_n) \\
&= \left[s_n y_n - s_m y_{m+1}\right] + \sum_{j=m+1}^{n-1} s_j (y_j - j_{j+1})
\end{aligned}$$

2.7.13 **(Dirichlet's Test)** Dirichlet's Test for convergense states that if the partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded and if (y_n) is a non-decreasing sequence with $\lim y_n = 0$, then the series $\sum_{n=1}^{\infty} x_n y_n$ converges.

(a) Let $M > 0$ be an upper bound for the partial sums of $\left|\sum_{n=1}^{\infty} x_n\right|$. Use the prior excercise to show that

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| \leq 2M|y_{m+1}|.$$

(b) Prove Dirichlet's Test.

(c) Show how the Alternating Series Test can be derived as a special case of Dirichlet's Test.

solution

(a) We first note that since $\sum |x_n|$ is bounded by M, s_j, the j^{th} partial sum of x_n, is less than M for any $j \leq n$. Using this fact, we can deduce the following:

$$\begin{aligned}
\left|\sum_{j=m+1}^{n} x_j y_j\right| &= \left|s_n y_n - s_m y_{m+1} + \sum_{j=m+1}^{n-1} s_j (y_j - y_{j+1}\right| \quad \text{(from the previous problem)} \\
&\leq |s_n y_n - s_m y_{m+1}| + \left|\sum_{j=m+1}^{n-1} s_j (y_j - y_{j+1}\right| \quad \text{by the triangle inequality} \\
&\leq |M y_n - M y_{m+1}| + \left|\sum_{j=m+1}^{n-1} M (y_j - y_{j+1}\right| \\
&\leq M\left(|y_n - y_{m+1}| + \left|\sum_{j=m+1}^{n-1} (y_j - y_{j+1}\right|\right)
\end{aligned}$$

Before we move on, let's look at the sum on the right in the last line. We can observe that all but two terms in this sum will cancel:

$$\left|\sum_{j=m+1}^{n-1}(y_j - y_{j+1})\right| = \left|y_{m+1} - \cancel{y_{m+2}} + \cancel{y_{m+2}} - \cancel{y_{m+3}} + \cdots + \cancel{y_{n-2}} - \cancel{y_{n-1}} + \cancel{y_{n-1}} - y_n\right|$$

Since every y_n is positive, it is also true that $|y_n - y_{m+1}| < |y_{m+1}|$, and is likewise true for $|y_{m+1} - y_n|$ Now continuing what we had before:

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| \le M\left(|y_n - y_{m+1}| + \left|\sum_{j=m+1}^{n-1}(y_j - y_{j+1})\right|\right) = M\left(|y+n - y_{m+1}| + |y_{m+1} - y_n|\right) < M \cdot 2|y_{m+1}|$$

(b) Now let $\epsilon > 0$ Since y_n is non-increasing, and converges to zero, we can choose N so that for $m+1 \ge N$, $y_{m+1} < \epsilon$. In particular, we can make $y_{m+1} < \frac{\epsilon}{2M}$. Using part (a), we have

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| < 2M|y_{m+1}| < 2M\frac{\epsilon}{2M} = \epsilon$$

.

(c) Since the partial sums of $(-1)^n$ are bounded (by any $M > 1$), it is easy to see that the alternating series test is a special case of Dirichlet's test.

Making some terms of a series negative Suppose (a_n) decreases monotonically to a limit of 0, and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(a) Does the series

$$a_1 + a_2 - a_3 + a_4 + a_5 - a_6 + a_7 + a_8 - a_9 \ldots$$

converge of diverge?

(b) Does the series

$$a_1 + a_2 - a_3 - a_4 + a_5 + a_6 - a_7 - a_8 + \ldots$$

converge or diverge?

(c) Generalize (a) and (b).

solution

1. Our claim is that this first series diverges. First, consider the grouping $a_1 + (a_2 - a_3) + a_4 + (a_5 - a_6) + a_7 + (a_8 - a_9)\cdots$. Since (a_n) is non-increasing, each of the terms in parentheses is either positive or zero, but never negative. Using this we can observe the following:

$$\begin{aligned} a_1 + (a_2 - a_3) + a_4 + (a_5 - a_6) + a_7 + (a_8 - a_9)\cdots &\ge a_1 + a_4 + a_7 + a_{10} + \cdots \\ &= \frac{1}{3}(3a_1 + 3a_4 + 3a_7 + 3a_{10}\cdots) \\ &\ge \frac{1}{3}(a_1 + a_2 + a_3 + a_4 + a_5 + \cdots) \end{aligned}$$

The last series is just $\frac{1}{3}\sum a_n$, which diverges, so the the origninal series must also diverge.

2. Consider the sequence $x_n = (1, 1, -1, -1, 1, 1, -1, -1, \dots)$. The partial sums of this series are bounded by $M > 2$. Using the fact that (a_n) is non-increasing with limit equal to zero, we can apply Dirichlet's test to conlude that $\sum x_n a_n = a_1 + a_2 - a_3 - a_4 + a_5 + a_6 \cdots$ converges.

3. Any sequence of the form $x_n = (\overbrace{1, 1, 1, 1, 1 \cdots}^{n_1\text{times}} \overbrace{-1, -1, -1, -1}^{n_1\text{times}} \cdots \overbrace{1, 1, 1, 1 \cdots}^{n_2\text{times}} \overbrace{-1, -1, -1, -1}^{n_2\text{times}} \cdots)$ will be bounded by $M > \max\{n_k\}$, so that Dirichlet's test can be applied to $\sum x_n a_n$.

3.2.9

(a) If y is a limit point of $A \cup B$, show that y is either a limit point of A, a limit point of B or of both

Proof. Suppose that y is a limit point of $A \cup B$. Then there is some sequence in $A \cup B$ that converges to y. Further, there is an ϵ-ball around y completely contained within either A or B (perhaps both) that contains an infinite number of members from the sequence converging to y. The sequence in this ϵ-ball shows that y is a limit point of either A or B (or perhaps both). $\square$

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. First, the forward inclusion. Let $x \in \overline{A \cup B}$. Then $x \in (A \cup B)' \cup (A \cup B) = (A \cup B)' \cup A \cup B$. Then consider the following:

- $x \in (A \cup B)' \implies x \in A' \cup B' \implies x \in (A' \cup A) \cup (B' \cup B) = \overline{A} \cup \overline{B}$. This was shown in part (a).
- $x \in A \Rightarrow x \in A \cup A' = \overline{A} \implies x \in \overline{A} \cup \overline{B}$
- If $x \in B$, then the argument is the same as $x \in A$.

This shows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Now for the backwards inclusion. Let $x \in \overline{A} \cup \overline{B}$. Then, again consider the following:

- $x \in \overline{A} \implies x \in A \cup A' \implies x \in (A \cup B)' \cup (A \cup B) = \overline{A \cup B}$
- $x \in \overline{B} \implies x \in B \cup B' \implies x \in (A \cup B)' \cup (A \cup B) = \overline{A \cup B}$

Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$, so the equality holds. $\square$

(c) Does the result about closures in (b) extend to infinite unions of sets?

3.2.13 Prove that the only sets that are both open and closed are $\mathbb{R}^n$ and $\emptyset$.

Proof. First assume that $A \subset \mathbb{R}$ is open, and that its complement $\mathbb{R} \backslash A$ is also open. Now let x_0 be some element of A, and define $B := \{y : [x_0, y] \subseteq A\}$. Let $s = \sup B$, and consider the following cases:

(i) *Case 1:* Suppose $s \in A$. Since A is open, $(s + \epsilon) \in A$ for some ϵ. Then there is some z, such that $s < z < s + \epsilon$. This means that $[x_0, z] \subseteq A$, which would imply that $z \in B$. This is a contradiction, since z could not be greater than s, the supremum of B.

(ii) *Case 2:* Now assume that s is in $\mathbb{R} \backslash A$. This set is also closed, so $(s - \epsilon) \in \mathbb{R} \backslash A$ as well. Then $s - \epsilon > y$ for all $[x_0, y] \subset A$. This also contradicts the definition of supremum, since s is less than or equal to every other upper bound for B.

This means that A cannot be bounded above. The exact same argument using infimums can show that A cannot be bounded below either. Since A is not bounded. abover or below, it either contains all real numbers or none of them. $\square$

Limits in $\mathbb{R}^2$: Prove that if (x_k, y_k) is a sequence in $\mathbb{R}^2$, then $\lim\limits_{k\to\infty}(x_k, y_k) = (a, b)$ iff $\lim\limits_{k\to\infty} x_k = a$ and $\lim\limits_{k\to\infty} y_k = b$.

Proof. ($\Longrightarrow$) First assume that $\lim\limits_{k\to\infty}(x_k, y_k) = (a, b)$. Then for all $\epsilon > 0$, we can find a K, such that for $k \geq K$, $\sqrt{(x_k - a)^2 + (y_k - b)^2} < \epsilon$. First we show that $\lim x_n = a$. To do this, it is important to note that

$$|x_k - a| = \sqrt{(x_k - a)^2} \leq \sqrt{(x_k - a)^2 + (y_k - b)^2}$$

Therefore, for the same K we used before is sufficient x_k for the same ϵ. The argument is identical to show that $\lim y_n = b$.

($\Longleftarrow$) Now assume that $\lim x_k = a$ and $\lim y_k = b$. Then for $k \geq K_1$, $|x_k - a| = \sqrt{(x_k - a)^2} < \frac{\epsilon}{2}$. Similarily, for $k \geq K_2$, $|y_k - b| = \sqrt{(y_k - b)^2} < \frac{\epsilon}{2}$. Choose $K' = \max\{K_1, K_2\}$. Then for $k \geq K'$,

$$\sqrt{(x_k - a)^2 + (y_k - b)^2} \leq \sqrt{(x_k - a)^2} + \sqrt{(y_k - b)^2} = |x_k - b| + |x_k - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

3.2.14 A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.

(a) Show that a closed interval $[a, b]$ is a G_δ set.

solution: $[a, b] = \bigcap\limits_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$

(b) Show that the half-open interval $(a, b]$ is both a G_δ and and F_σ set.

solution:

- $(a, b] = \bigcap\limits_{n=1}^{\infty}(a, b + \frac{1}{n})$
- $(a, b] = \bigcup\limits_{n=1}^{\infty}[a + \frac{1}{n}, b]$

(c) Show that $\mathbb{Q}$ is an F_σ set, and that the set of irrationals, $\mathbb{I}$ forms a G_δ set.

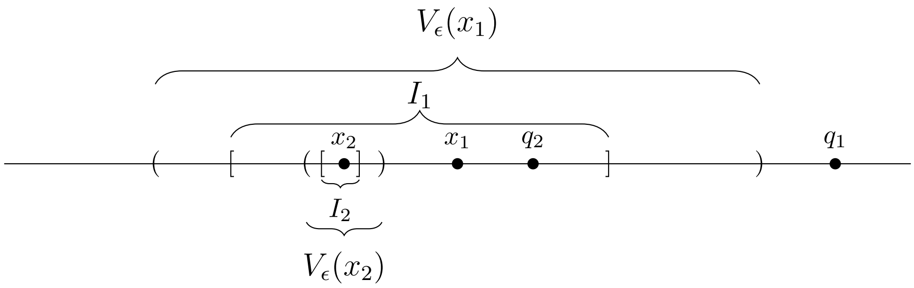
solution: $\mathbb{Q}$ is a countably infinite set[3], and singelton sets are closed, so $\mathbb{Q} = \bigcup\limits_{q\in\mathbb{Q}}\{q\}$. Then consider the sets formed by taking the complement of the rational singleton sets. Since $\{q\}$ is closed, $\mathbb{R}\backslash\{q\}$ is open. Further, $\mathbb{I} = \bigcap\limits_{q\in\mathbb{Q}}(\mathbb{R}\backslash\{q\})$, so $\mathbb{I}$ is an G_δ set.t

Rationals just wanna have fun Prove that $\mathbb{Q}$ is not a G_δ set.

Proof. Suppose that $\mathbb{Q} = \bigcap\limits_{k=1}^{\infty} U_k$, where each U_k is open. Consider the set $U_1\backslash\{q_1\}$, which contains every rational number except q_1. Pick a rational number x_1 in U_1 not equal to q_1. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find an ϵ-ball, $V_\epsilon(x_1)$, around x_1 such that $V_\epsilon(x_1) \cap \{q_1\} = \emptyset$. Next, we choose a closed interval inside $V_\epsilon(x_1)$. Now consider $U_2\backslash\{q_2\}$. Again, choose a rational number $x_2 \neq q_2$ that lies in U_2. There is a $V_\epsilon(x_2)$ that is disjoint from q_2,

[3] I prefer the term listably infinite, though it isn't used much.

allowing us to choose another closed interval. We can repeat this process indefinitely, giving us a nested chain of closed intervals ,such that $I_n \subseteq U_1 \cap U_2 \cap \cdots \cap U_n \cap I_{n-1} \backslash \{q_n\}$. By the nested interval theorem, there is an x^* that is in every I_n. Further, this x^* is not a rational number since it is in the intersection of sets that have had all the ratonals removed. But $x^* \in \bigcap_{k=1}^{\infty} U_k = \mathbb{Q}$. This means that x^* is an irrational number in $\mathbb{Q}$. This is a contradiction, so $\mathbb{Q}$ is not a G_δ set.



□

3.3.5 Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbb{Q}$

$\mathbb{Q}$ is not compact. Choose a sequence that converges to $\pi \notin \mathbb{Q}$.

(b) $\mathbb{Q} \cap [0, 1]$

This is the same as $\mathbb{Q}$ itself, instead we must choose some irrational number in $[0, 1]$ as a limit point. Choose a sequence that converges to $\frac{\pi}{4}$

(c) $\mathbb{R}$

We know that compact sets are bounded, so $\mathbb{R}$ cannot be compact. Choose any sequence that diverges to infinity, $an = n$

(d) $\mathbb{Z} \cap [0, 10]$

This set is closed and bounded, so by the Heine-Borel theorem it is compact.

(e) $W = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

The sequence $an = \frac{1}{n}$ in W has a limit equal to 0, which is not in the set.

(f) $T = \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$

It is easy to see that all the members of T are less than or equal to 1. It is also to see that each member is greater than or equal to $\frac{1}{2}$. This shows that the set is both closed and bounded, and therefore compact.

3.3.8 Assume that $K \subseteq \mathbb{R}$ is squentially compact, closed, and bounded. Prove that any open cover for K has a finite subcover.

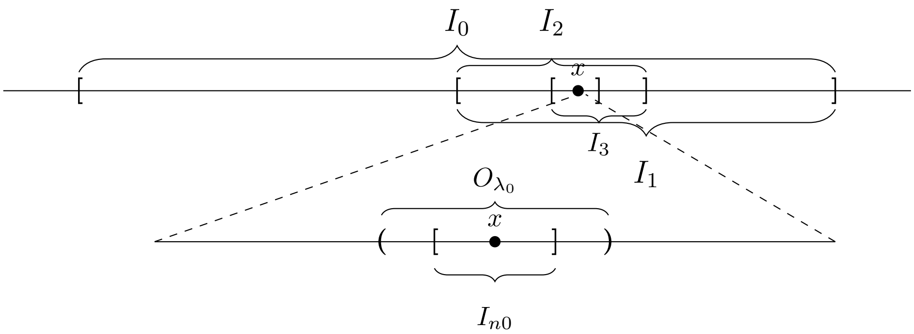
Proof. This will be a proof by contradiction, so assume that $\Omega = \{O_\lambda : \lambda \in \Lambda\}$, an open cover for K, has no finite subcover. I will follow the course that the book suggests.

(a) First let I_0 be a closed set such that $I_0 \supseteq K$, and partion I_0 into two closed sets, A and B. Let $K_1 = A \cap K$, and $K_2 = B \cap K$. Now suppose for the sake of contradiction that K_1 and K_2 have finite subcovers,$\{O_{x1}, O_{x2}, \ldots, O_{xn}\}$ and$\{O_{y_1}, O_{y_2}, \ldots, O_{y_m}\}$, respectively.

$$\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\} \cup \{O_{y_1}, O_{y_2}, \ldots, O_{y_m}\} \supseteq K_1 \cup K_2 = K$$

This suggests that K has a finite subcover, namely the union on the left side of the expression above. This is a contradiction, so $B \cap K$ has no finite cover.

(b) Take the interval I_0 described above and cut it in half into two closed sets. By part (a), at least one of these two closed sets when intersected with K will not have a finite subcover. Pick this interval, call it I_1, and divide it into two closed intervals. By the same logic as before, one of the two intervals intersected with K has no finite cover, call it I_2. We can repeat this process, creating a nest of intervals $I_0 \supset I_1 \supset I_2 \cdots \supset I_n \cdots$. Every interval is closed and bounded, so every interval is compact. Therefore, using theorem 3.3.5, there is a point x common to all the I_n. Further, since x is in every one of the I_n, it is also in K. Every element of K is contained in of the $O \in \{O_\lambda\}$, so x is in some $O_{\lambda 0}$.



It is fairly obvious that the limit of the lengths of the I_n is zero since the length of each $I_n = L\frac{1}{2^{n-1}}$. For every $\epsilon > 0$ there is some N, such that for $n \geq N$, $|I_n| < \epsilon$. Since x is in the open interval $O_{\lambda 0}$, there is some interval $(x - \epsilon, x + \epsilon)$ around x contained in $O_{\lambda 0}$. Since we can make $|I_n|$ arbitrarily small, we can find an n_0 such that $I_{n0} \subset (x - \epsilon, x + \epsilon)$. Then I_{n0} has a finite subcover $(O_{\lambda 0})$. This is a contradiction, since each I_n was assumed to lack that very property. Therefore K must have a finite open cover.

□

3.3.10 Describe all clompact spaces in $\mathbb{R}$.

Claim: K is clompact $\iff$ K is finite. Let $K = \{x_1, x_2, x_3, \ldots, x_n\}$ be a finite set. Also, let $\Lambda = \{O_\lambda\}$ be a closed cover for K. Then each x_k is in some O_k. Then the union of the O_k form a closed cover for K.

Now suppose K is clompact. Since singleton sets are closed, then $\bigcup\{x\} \subset K$ forms a closed cover for K. This closed cover must admit a finite subcover $\{x_1, x_2, \ldots, x_j\}$ by definition of clompact. Therefore K must be finite, since $K \subseteq \bigcup\{x_j\}$.

4.3.7 Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Proof. Let x^* be a limit point of K. We will show that x^* is in K. Since x^* is a limit point, there is a sequence $(x_n) \subseteq K$ with $(x_n) \to x^*$. Note that $f(x_n) = 0$ for each x_n, since each x_n is a member of K. Further, since h is continuous and $(x_n) \to x^*$, then $f(x_n) \to f(x^*)$. But since each $f(x_n) = 0$, then the sequence $f(x_n)$ must converge to zero implying that $f(x^*) = 0$, and thus $x^* \in K$.

□

4.3.9 (Contraction Mapping Theorem) Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

Proof. Let $\epsilon > 0$ be given. Define $\delta := \frac{\epsilon}{c}$. Then if $|x - y| < \delta$, then $|f(x) - f(y)| < c|x - y| < c\left(\frac{\epsilon}{c}\right) = \epsilon$. This means f is continuous for any $y \in \mathbb{R}$. □

(b) Pick some point $y_1 \in \mathbb{R}$ and constuct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots)$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

Proof. Let $\epsilon > 0$ be given. We want to show that for some N, $m, n \geq N$ implies that $|y_m - y_n| < \epsilon$. First, we rewrite the sequence as a telescoping series and apply the triangle inequality:

$$|y_m - y_n| \quad = \quad \left|\sum_{j=n+1}^{m} (y_j - y_{j-1})\right| \leq \sum_{j=n+1}^{m} \left|y_j - y_{j-1}\right|$$

Before we can work with the expression on the right, it is important to note the following observation of the property of f: $|y_3 - y_2| = |f(f(y_1)) - f(y_1)| < c|f(y_1) - y_1| = c|y_2 - y_1|$. This can be extended further:

$$|y_j - y_{j-1}| < c|y_{j-1} - y_{j-2}| < c \cdot c|y_{j-2} - y_{j-3}| < \dots < c^{j-2}|y_2 - y_1|$$

This means that we can rewrite the expression $\sum_{j=n+1}^{m} \left|y_j - y_{j-1}\right|$ in terms of powers of c multiplied by $|y_2 - y_1|$:

$$\sum_{j=n+1}^{m} \left|y_j - y_{j-1}\right| \quad < \quad \sum_{j=n-1}^{m-2} c^j |y_2 - y_1| = c^{n-1}|y_2 - y_1| \sum_{j=0}^{n-1} c^j < \frac{c^{n-1}|y_2 - y_1|}{1 - c}$$

The rightmost inequality is valid because $\sum c^j$ is a geometric series. Further, because $0 < c < 1$, we can make c^{n-1} arbitralily small. This means we can find an N such that for $n - 1 > N$, $c^{n-1} < \frac{\epsilon(1-c)}{|y_2 - y_1|}$. Now for $m, n \geq N + 1$ the following is true:

$$|y_m - y_n| \quad \leq \quad \sum_{j=n+1}^{m} \left|y_j - y_{j-1}\right| < \frac{c^{n-1}|y_2 - y_1|}{1 - c} < \frac{c^{n-1}|y_2 - y_1|}{(1 - c)} \cdot \frac{(1 - c)}{c^{n-1}|y_2 - y_1|} = \epsilon$$

Thus the defined sequence is a Cauchy sequence and therefore converges to a limit we call y. □

(c) Prove that y is a fixed point of f and that it is unique in this regard.

Proof. We want to show that $f(y) = y = \lim y_n$. In (a), it was shown that f is continuous over the reals, so $y_n \to y$ implies $f(y_n) \to f(y)$. It is also true that $y_{n+1} \to y$, so $f(y_{n+1}) \to f(y)$. But $f(y_{n+1}) = y_n$, by

definition of the sequence. Stating this again,

$$y_{n+1} \to y \implies f(y_{n+1}) = y_n \to f(y)$$

We have $y_n \to y$ and $y_n \to f(y)$, and since limits are unique, it must be true that $y = f(y)$. □

(d) Finally, prove that if x_0 is any arbitrary point in $\mathbb{R}$, then the sequence $(x_n) = x_0, f(x_0), f(f(x_0)), \ldots$ converges to y defined in (b).

Proof. Let x_0 be some point in $\mathbb{R}$, and let $x_n \to x$. From part (c), it also must be true that $f(x) = \lim x_n = x$. We want to show that $x = y$. To do so, we take note of the contraction mapping property (is that correct wording?) that $|f(x) - f(y)| \leq c|x - y|$. But $|f(x) - f(y)| = |x - y|$. Now we can rewrite the prior expression as $|f(x) - f(y)| = |x - y| \leq c|x - y|$. We have $0 < c < 1$, though, so the only way this is possible is if both sides are zero, or in other words, $|x - y| = 0 \implies x = y$.

□

4.3.10 Let f be a function defined on all of $\mathbb{R}$ such that f is additive over the reals.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$.

Proof. First we show that f maps zero to zero.

$$\begin{aligned} f(x) &= f(x+0) = f(x) + f(0) \\ f(x) &= f(x) + f(0) \\ 0 &= f(0) \end{aligned}$$

Now we show that $f(-x) = -f(x)$

$$\begin{aligned} f(0) &= f(x + (-x)) = f(x) + f(-x) \\ 0 &= f(x) + f(-x) \\ -f(x) &= f(-x) \end{aligned}$$

□

(b) Show that if f is continuous at $x = 0$ then f is continous at every point in $\mathbb{R}$.

Proof. Assume that if $(x_n) \to 0$, then $f(x_n) \to f(0) = 0$ (that is that f is continuous as $x = 0$). We need to show that if $(a_n) \to c$, then $f(a_n) \to f(c)$. Since $(a_n) \to c$, then $(a_n - c) \to c - c = 0$. By hypothesis, this means that $f(a_n - c) = f(0) = 0$. Since f is linear, this means that $f(a_n) - f(c) = 0$ or $f(a_n) = f(c)$. Thus f is continuous at c, and since c was arbitrary, f is continuous as all of $\mathbb{R}$. □

(c) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$. Then do the same for $\mathbb{Z}$ and $\mathbb{Q}$.

Proof. Let $n \in \mathbb{N}$. Then $f(n) = f(\overbrace{1 + 1 + \cdots + 1}^{\text{n times}}) = \overbrace{f(1) + f(1) + \cdots + f(1)}^{\text{n times}} = nk$.
The argument is identical to show $f(z) = zn$, $z \in \mathbb{Z}$, except that we need to mention the fact that

$f(-x) = -f(x)$ in the case where z is odd. Now let $r = \frac{m}{n}$ (in reduced form) be a rational number. Then

$$\begin{aligned}
mk = f(m) = f(n\frac{m}{n}) &= f(\overbrace{\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}}^{\text{n times}}) = nf\left(\frac{m}{n}\right) \\
mk &= nf\left(\frac{m}{n}\right) \\
k\left(\frac{m}{n}\right) &= f\left(\frac{m}{n}\right) \\
kr &= f(r)
\end{aligned}$$

□

(d) Use (b) and (c) to conlude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus any additive function that is continuous at $x = 0$ must necessarily be a linear function through the orgin.

Proof. Let x be some element of $\mathbb{R}$. Let us choose a sequence in $\mathbb{R}$ that converges to x. It will be useful to choose a sequence with the property that $f(x_n) = kx_n$, which is easy since we can just choose a sequence from the rational numbers to converge to x. Then, by continuity, $x_n \to x \implies f(x_n) \to f(x)$. But $f(x_n) = kx_n$ for all n, so $f(x_n) \to f(x) \implies kx_n \to f(x)$. But if we have $kx_n \to kx$ and $kx_n \to f(x)$, then by the uniqueness of limits, $f(x) = kx$.

□

4.4.2 Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

First we show that f is uniformly continuous on $[1, 0)$. First we look at $|f(x) - f(a)|$:

$$\begin{aligned}
|f(x) - f(a)| &= \left|\frac{1}{x^2} - \frac{1}{a^2}\right| \\
|f(x) - f(a)| &= \frac{|a^2 - x^2|}{(ax)^2} \\
|f(x) - f(a)| &= \frac{|a - x| \cdot |a + x|}{(ax)^2} \\
|f(x) - f(a)| &= \frac{x + a}{(ax)^2}|a - x| \\
|f(x) - f(a)| &= \left(\frac{x}{(ax)^2} + \frac{a}{(ax)^2}\right)|a - x| \\
|f(x) - f(a)| &= \left(\frac{1}{a^2x} + \frac{1}{ax^2}\right)|x - a|
\end{aligned}$$

The term $\left(\frac{1}{ax^2} + \frac{1}{a^2x}\right)$ is less than 2, since each component is less than 1 (because our domain restricts us to x greater than or equal to 1). This provides us with a suitable choice for δ.

Proof. Let $\epsilon > 0$ be given, and define $\delta := \frac{\epsilon}{2}$. Then,

$$|x-a| < \delta \Longrightarrow |f(x) - f(a)| = \left(\frac{1}{ax^2} + \frac{1}{a^2x}\right)|x-a| < 2\left(\frac{\epsilon}{2}\right) = \epsilon$$

□

Now we show that f is *not* continuous on $(0,1]$.

Proof. Let $\epsilon = 1$, and $(x_n) = \frac{1}{n} \subseteq (0,1]$ and $(y_n) = \frac{1}{n+1} \subseteq (0,1]$. Then $|x_n - y_n| = \frac{1}{n(n+1)} \to 0$. However, $|f(x_n) - f(y_n)| = |-2n-1| = |2n+1| \geq 2 > 1 = \epsilon$. Therefore, f is not uniformly continuous on $(0,1]$. □

To be uniformly continuous or not to be uniformly continuous, that is the question. Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x^2}$ uniformly continuous.

Proof. We intend to show that this function is uniformly continuous. First, assume that we have to sequences (x_n) ,$(y_n) \subseteq \mathbb{R}$ with the property that $|x_n - y_n| \to 0$. To show that f is uniformly continuous, it is necassary to show that $|f(x_n) - f(y_n)| \to 0$ as well. To do this, we start by writing $|f(x_n) - f(y_n)|$:

$$\begin{aligned}
|f(x_n) - f(y_n)| &= \left|\frac{1}{1+x_n^2} - \frac{1}{1+y_n^2}\right| \\
|f(x_n) - f(y_n)| &= \left|\frac{y_n^2 - x_n^2}{(1+x_n^2)(1+y_n^2)}\right| \\
|f(x_n) - f(y_n)| &= \left|\frac{(y_n - x_n)(y_n + x_n)}{(1+x_n^2)(1+y_n^2)}\right| \\
|f(x_n) - f(y_n)| &= |(y_n - x_n)|\left|\frac{y_n + x_n}{(1+x_n^2)(1+y_n^2)}\right| \\
|f(x_n) - f(y_n)| &\leq |(y_n - x_n)|\left(\left|\frac{y_n}{(1+x_n^2)(1+y_n^2)}\right| + \left|\frac{x_n}{(1+x_n^2)(1+y_n^2)}\right|\right) \\
|f(x_n) - f(y_n)| &< |(y_n - x_n)|\left(\left|\frac{y_n}{(1+y_n^2)}\right| + \left|\frac{x_n}{(1+x_n^2)}\right|\right)
\end{aligned}$$

Each of the sequences in the parentheses are bounded by any $M \geq \frac{1}{2}$. Here is a proof for the sequence with terms from y_n.

$$\begin{aligned}
(y_n - 1)^2 &\geq 0 \\
y_n^2 - 2y_n - 1 &\geq 0 \\
y_n^2 + 1 &\geq 2y_n \\
\frac{1}{2} &\geq \frac{y_n}{1+y_n^2}
\end{aligned}$$

Therefore we can conlude the following:

$$|f(x_n) - f(y_n)| < |y_n - x_n|\left(\left|\frac{y_n}{(1+y_n^2)}\right| + \left|\frac{x_n}{(1+x_n^2)}\right|\right) \leq |y_n - x_n|(\frac{1}{2} + \frac{1}{2}) = |y_n - x_n|$$

By hypothesis $|y_n - x_n| \to 0$, so $|f(x_n) - f(y_n)| \to 0$ as well, so f is uniformly continuous over the reals. □

4.4.6 Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case?

(a) a continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Suppose that we have a sequence (x_n) which converges to 0. Then $f(x) := \dfrac{1}{x_n}$ is continuous and will be unbounded, won't converge, and thus not be Cauchy.

(b) a continuous function $f : [0, 1] \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

This is impossible. Let $(x_n) \subseteq [0, 1]$ be a Cauchy sequence. Then (x_n) converges to some point c in $[0, 1]$ (because $[0, 1]$ is closed). Since f is assumed to be continuous, if $x_n \to c$ then $f(x_n) \to f(c)$. Therefore $f(x_n)$ must converge, so is also Cauchy.

(c) a continuous function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

This is impossible for the exact same reason as the last. Since $\mathcal{D}(f)$ is closed, it contains its limits points. Again, because of continuity, the image of the sequence must converge.

(d) a continuous bounded function f on $(0, 1)$ that attains a maximum value on this open interval but not a minimum value.

A good choice is a polynomial which has a vertex whose x component is in the interval $(0, 1)$. We want a polynomial of the form $p(x) = -(x - a)^{\frac{1}{2}} + b$. To construct a specific polynomial, let us choose one with roots at 0 and 1, so that the function won't take on a minimum value. This requires that $p(0) = p(1) = 0$. If $p(0) = p(1)$, then $\cancel{-a^2} = -1 + 2a - \cancel{a^2} \implies a = \frac{1}{2}$. Further if $p(0) = \frac{-1}{4} + b = 0$, then $b = \frac{1}{4}$. Therefore the polynomial $p(x) = -(x - \frac{1}{2})^2 + \frac{1}{4}$ has a maximum, but will never attain a minimum.

Partial converse to Theorem 4.4.3. Let $K \subseteq \mathbb{R}$. Suppose that for all continuous functions $f : K \to \mathbb{R}$, $f(K)$ is bounded. Then K is compact.

Proof. This is will be a proof by contrapositive. Assume that K is not compact. Then we will show that there is a continuous function $f : K \to \mathbb{R}$ such that $f(K)$ is unbounded. Since K is not compact, it is true that K is not closed or it is not bounded (perhaps both).

(i) *Case 1*: K is not closed. Then there is some sequence (x_n) in K that converges to a point c not in K. Then the function $f(x) := \dfrac{1}{x - c}$, which is continuous, satisfies the condition that $f(K)$ is unbounded.

(ii) *Case 2*: K is not bounded. Then the function $f(x) := x$ is continuous and $f(K)$ is unbounded.

$\square$

5.2.8. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If a derivative function is not constant, then the derivative must take on some irrational values.

This is not necassarily true. Just consider the absolute value function. It's derivative is not constant over $\mathbb{R}$, but never takes on an irrational value. The function defined by $f(x) = nx - \dfrac{n(n-1)}{2}$, for $x \in [n-1, n)$ is continuous everyhwere in its domain, differentiable almost everywhere (though only on open intervals), but it's derivative is never irrational. If we make a stronger requirement, that f' is not constant on a closed interval, then this claim *is* true using Darboux's theorem.

(b) If f' exists on an open interval, and there is some point c where $f'(c) > 0$ then there exists a $\delta-$neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$

This is false. This is the characterisitic of a continuous function. We have shown that a deriviative need not be continuous.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Proof. This will be a proof by contradiction. Assume that f is differentiable on an interval containing zero, with $\lim_{x\to 0} f'(x) = L$, but $f'(0) = M > L$. Since for every $V_\epsilon(L)$, we can find a $V_\delta(0)$, such that $x \in V_\delta(0)$ (and $x \neq 0$) $\implies f'(x) \in V_\epsilon(L)$. Now choose an $\epsilon = \dfrac{M-L}{2} = \dfrac{f'(0)-L}{2}$, so that $V_\epsilon(L)$ misses $f'(0)$. Now consider the closed interval $[x^*, 0]$, where $x^* \in V_\delta(0)$. Then we can choose an α with the following property:

$$f'(x^*) < L + \epsilon < \alpha < f'(0)$$

Then by Darboux's theorem, there exists a $c \in [x^*, 0]$ such that $f'(c) = \alpha$. But we chose ϵ so that for all x not equal to zero, there is a space between $V_\epsilon(L)$ and $f'(0)$, which means that $f'(x)$ cannot be greater than balh. $\square$

(d) Repeat conjecture (c) but drop the assumption that $f'(0)$ necessarily exists.

Proof. An important note before the proof. To use the mean value theorem, I believe we need to add that f is continuous at $x = 0$.

Now we have $f'(x) \to L$ at $x = 0$, and f is continuous on $[0, a]$ and differentiable over $(0, a)$, for some $a > 0$. Then by the mean value theorem, there is a $c_a \in (0, a)$ with the property that $f'(c_a) = \dfrac{f(a) - f(0)}{a}$. We can find such a c_a for each open interval $(0, a)$, so if we take the limit as a approaches zero, we have the following.

$$\lim_{a\to 0} f'(c_a) = \lim_{a\to 0} \frac{f(a) - f(0)}{a} = f'(0)$$

But as $a \to 0$, $c_a \to 0$, so $\lim_{a\to 0} f'(c_a) = \lim_{c_a\to 0} f'(c_a) = L$. Thus, bringing it all together, we have:

$$\lim_{a\to 0} f'(c_a) = \lim_{c_a\to 0} f'(c_a) = L = f'(0)$$

$\square$

5.3.6 Let $g : [0, 1] \to \mathbb{R}$ be twice-differentiable with $g''(x) > 0$ for all $x \in [0, 1]$. If $g(0) > 0$ and $g(1) = 1$, show that $g(d) = d$ for some point $d \in (0, 1)$ if and only if $g'(1) > 1$.

Proof. ($\Longrightarrow$) First assume that there is some fixed point d in $(0,1)$. Since g is continuous on $[0,1]$, g is also continuous on $[d,1]$ Applying the mean value theorem, there is a $c_1 \in (d,1)$ such that

$$g'(c_1) = \frac{g(1)-g(d)}{1-d} = \frac{1-d}{1-d} = 1$$

Let's use c_1 and apply the mean value theorem again, but this time to g', so that there is a $c_2 \in (c_1,1)$, such that

$$g''(c_2) = \frac{g'(1)-g'(c_1)}{1-c_1} = \frac{g'(1)-1}{1-c_1}$$

The quantity in the denominator is positive, and by hypothesis, $g''(c_2) > 0$, so looking at the numerator, $g'(1) > 1$.

□

5.3.7

Proof. ($\Longrightarrow$) Assume that f is increasing. Then $\dfrac{f(x)-f(y)}{x-y} \geq 0$, for any arbitrary $x, y \in (a,b)$, $x \neq y$. Since this is greater than or equal to 0 for any choice of x and y, then $\lim\limits_{x\to y} \dfrac{f(x)-f(y)}{x-y} \geq 0$. But this limit is $f'(x)$, so we are done.

($\Longleftarrow$). Now assume that

$$f'(x) = \lim_{x\to y} \frac{f(x)-f(y)}{x-y} \geq 0$$

We can use the mean value theorem to conclude that for some value of c:

$$f'(c) = \frac{f(y)-f(x)}{y-x} \geq 0$$

If $x > y$, then we must have $f(y) - f(x) \geq 0 \Longrightarrow f(y) > f(x) \Longrightarrow f$ is increasing. The argument is identical for the case in which $x > y$ to show that f is decreasing.

The derivative at zero is:

$$f'(0) = \lim_{x\to 0} \frac{\frac{x}{2} + x^2 \sin(\frac{1}{x}) - 0}{x-0} = \lim_{x\to 0} \frac{1}{2} + x\sin(\frac{1}{x}) = \frac{1}{2} > 0$$

For any other value of x however, we have $g'(x) = \dfrac{1}{2} - \cos\left(\dfrac{a}{x}\right) + 2x\sin\left(\dfrac{1}{x}\right)$. Now consider the sequence $\delta_n = \dfrac{1}{1+2n\pi}$. Then $V_{\delta_{n+1}}(0) \subset V_{\delta_n}(0)$ for all n. Therefore, $f'(x)$ cannot be increasing over any $V_{\delta_n}(0)$, because if we choose $\dfrac{1}{1+(n+1)\pi} \in V_{\delta_{n+1}}(0)$, then $f'\left(\dfrac{1}{1+(n+1)\pi}\right) < 0$. Since $\delta_n \to 0$, there must be no open interval that is increasing for all x in that interval.

□

5.3.8

Proof. If $g'(c) \neq 0$ then $g'(c) > 0$ or $g'(c) < 0$. Let's consider the first case, and choose ϵ so that $\epsilon < g'(c)$. Then consider the ϵ-neighborhood $V_\epsilon(g'(c))$. By continuity, there is a $\delta-$neighborhood such that for $x \in V_\delta(c)$, $g(x) \in V_\epsilon(g'(c)) > 0$ (by our choice of ϵ) . If $x < c$, then

$$\lim_{x\to c} \frac{g(x)-g(c)}{x-c} > 0 \Longrightarrow g(x) > g(c)$$

The argument is similar for $x > c$ to show $g(x) < g(c)$. Thus for all $x \in V_\delta(c)$, with $x \neq c$, $g(x) \neq g(c)$.

The proof is nearly identical for the case in which $g'(c) < 0$. □

5.4.5 To show that $g'(1)$ does not exist, let $x_m = 1 + \frac{1}{2^m}$. Then:

$$g(x_m) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n + 2^{n-m})$$

For $n = 0$, we have $\frac{1}{2^0} h(1 + \frac{1}{2^m}) = 1 - \frac{1}{2^m}$.

For $1 \leq n \leq m$, we have $\frac{1}{2^n} h(2^n + 2^{n-m} = \frac{1}{2^n} h(2^m - n) = \frac{1}{2^n} 2^{n-m} = \frac{1}{2^m}$. This does not depend on n, so we will have $m - 1 \frac{1}{2^m}$ terms

For $n > m$, each term will be zero, since 2 will divide the term in $h(x)$.

From this, we can look at $g'(1)$.

$$g'(1) = \lim_{x \to 1} \frac{g(x_m) - g(1)}{x_m - 1} = \lim_{x \to 1} \frac{1 + (m-1)2^{-m} - 1}{\frac{1}{2^m}} = m - 1$$

As $m \to \infty$, $g'(1) \to \infty$, so $g'(1)$ does not exist. Hooray!

Now to show that g is not differentiable at $x = \frac{1}{2}$. Similar to before, we set $x_m = \frac{1}{2} + \frac{1}{m^2}$. First we compute $g(x_m)$:

$$\begin{aligned} g(\frac{1}{2} + \frac{1}{2^m}) &= \sum_{n=0}^{1} \frac{1}{2^n} h(2^{n-1} + 2^{n-m}) + \sum_{n=2}^{m} \frac{1}{2^n} h(2^{n-1} + 2^{n-m}) \\ g(\frac{1}{2} + \frac{1}{2^m}) &= (\frac{1}{2} + \frac{1}{2^m}) + (1 - \frac{1}{2^{m-1}}) + (m-2)\frac{1}{2^m} - g(\frac{1}{2}) \end{aligned}$$

Simplifying and looking at the derivative at $\frac{1}{2}$

$$g'(\frac{1}{2}) = \lim_{x_m \to \frac{1}{2}} \frac{\frac{3}{2} + \frac{1}{2^m} - \frac{1}{2^{m-1}} + (m-2)\frac{1}{2^m} - \frac{3}{2}}{\frac{1}{2^m}} = (m - 4)$$

Again, this diverges as m grows large, so we are done.

5.4.6

(a) Since each h_i is differentiable at non-dyadic points, then it follows that the partial sums of g_m are also differentiable at non-dyadic points. Now observe that:

$$|g'_{m+1}(x) - g'_m(x)| = |(h_0 + h_1 + \dots h_m + h_{m+1}) - (h_0 + h_1 + \dots h_m)| = |h_{m+1}(x)| = |\pm 1| = 1$$

(b) ?????????

(c) From part (a) we can note that there is no possible way, given $\epsilon > 1$, to choose an N, so that for $n, m \geq N$ it follows that $|g'_n(x) - g'_m(x)|$. This means that $g'(x)$ cannot converge. From part (b),if we take the limit as $m \to \infty$:

$$\lim_{m \to \infty} \frac{g(y_m) - g(x)}{y_m - x} < \lim_{m \to \infty} g'_m(x) < \lim_{m \to \infty} \frac{g(x_m) - g(x)}{x_m - x}$$

or in other words:

$$g'(x) < \lim_{m\to\infty} g'_m(x) < g'(x)$$

But by the squeezey theorem, this would imply that $g'(x)$ is equal to the limit in the middle, which we proved does not exist. Therefore, there g must not be differentiable.

5.4.7 If we look at part (a) of problem 5.4.6, we can see that $|g'_{m+1}(x) - g'_m(x)|$ will give values that increase without bound as m approaches infinity, so that $|g'_{m+1}(x) - g'_m(x)|$ does not converge. For the second example however, $|g'_{m+1}(x) - g'_m(x)|$ converge to 0 as m approaches infinity. Therefore, we cannot assume that g is not differentiable at non-dyadic points.

6.2.1 Let

$$f_n(x) = \frac{nx}{1+nx^2}$$

(a) Find the pointwise limit of (f_n) for all $x \in (0,\infty)$.

answer The limit of this series of functions is

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+nx^2} = \frac{1}{x}$$

(b) Is the convergence uniform on $(0,\infty)$?

answer No. Consider $|f_n(x) - f(x)|$:

$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \left|\frac{nx^2-1-nx^2}{x(1+nx^2)}\right| = \left|\frac{1}{x+nx^3}\right|$$

Given an $\epsilon > 0$, we would have to choose $N > \dfrac{1-\epsilon x}{\epsilon x^3}$ to satisfy convergence. The fact that this choice is dependent on x tells us that the convergence is not uniform.

(c) On $(0,1)$?

answer No. In fact, any interval containing zero will not be uniformly convergent because the dependence on x becomes unruly when we look at x that are close to 0.

(d) $(1,\infty)$?

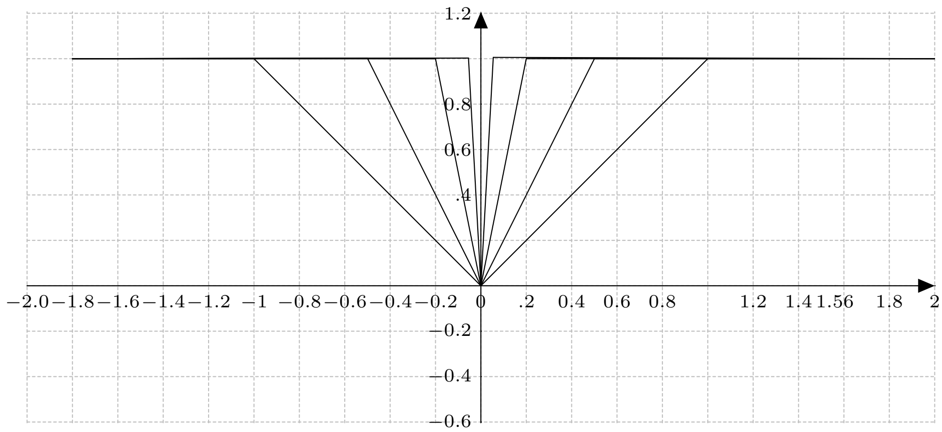
answer Yes. Because given $x > 1$, we can say

$$\frac{1}{x+nx^3} < \frac{1}{1+n}$$

This means that our choice of N need not depend on x, so the convergence over this domain is uniform.

6.2.5 For each $n \in \mathbb{N}$, define f_n on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & : |x| \geq 1/n \\ n|x| & : |x| < 1/n \end{cases}$$



(a) Find the pointwise limit of (f_n) on $\mathbb{R}$ and decide whether or not the convergence is uniform.

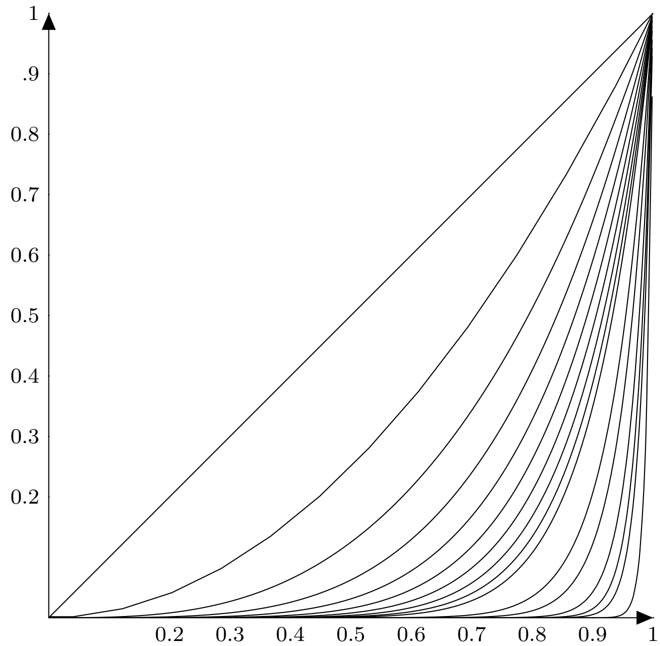
answer Given any $x \neq 0$, it is easy to see that $f_n(x)$ will eventually will equal 1, and stay there indefinitely. So the point wise limit on $(-\infty, 0) \cup (0, \infty)$ is 1. At $x = 0$, $f_n(0) = 0$ for all $n \in \mathbb{N}$, the here the pointwise limit is 0. The convergence would be uniform if it did not contain zero.

(b) Construct an example of a pointwise limit of continuous functions that converges everywhere on the compact set $[-5, 5]$ to a limit function that is unbounded on this set.

6.2.8 Decide which of the following conjectures are true and which are false. Suplly a proof for those that are valid and a counterexample for each one that is not.

(a) If $f_n \to f$ pointwise on a compact set K, then $f_n \rightrightarrows f$ on K.

answer False. The simplest example is (f_n), where $f_n(x) = x^n$ over $[0, 1]$, which we showed in class to not be uniformly convergent.



(b) If $f_n \rightrightarrows f$ on A and g is a bounded function on A, then $f_n g \rightrightarrows fg$ on A.

*answer:*True

Proof. Suppose that $f_n \rightrightarrows f$ and that $g(x)$ is bounded. Since $g(x)$ is bounded, there is an M such that $|g(x)| < M$ for all $x \in \mathcal{D}(g)$. Since f_n converges uniformly, given an $\epsilon > 0$, there is an N so that $n \geq N$

implies $|f_n(x) - f(x)| < \epsilon$. We are free to choose N so that $|f_n - f|$ is less than $\dfrac{\epsilon}{M}$. Therefore, we can use the same N to conclude:

$$|f_n g - gf| = |g| \cdot |f_n - f| < M\frac{\epsilon}{M} = \epsilon$$

. Since our original choice for N did not depend on x, this convergence is uniform. □

(c) If $f_n \rightrightarrows f$, and if each f_n is bounded on A, then f must also be bounded.

answer: True

Proof. Let $\epsilon > 0$ be given. Since $f_n \rightrightarrows f$, we can make $|f_n(x) - f| < \epsilon$ for all n greater than some N. f_n is bounded by some number M by hypothesis. Since $|f_n - f| < \epsilon$ for $n \geq N$, each subsequent $f_n(x)$ after $f_N(x)$ will also be bounded by M. It follows directly from the fact that $|f_n - f| < \epsilon$ that f is bounded by $\epsilon + M$, so the proof is done. □

(d) If $f_n \rightrightarrows f$ on set A, and if $f_n \rightrightarrows f$ on a set B, then $f_n \rightrightarrows f$ on $A \cup B$.

answer: True

Proof. By hypothesis, we can choose N_1, so that $|f_n(x) - f(x)| < \epsilon$, for all $\epsilon > 0$, $x \in A$ and $n \geq N_1$. We can choose an N_2 for B in the same manner. If we choose $N^* = \max\{N_1, N_2\}$, then it is clear that $|f_n(x) - f(x)|$ will be less than ϵ for all $x \in A \cup B$ when $n \geq N^*$. □

(e) If $f_n \rightrightarrows f$ on an interval, and if each f_n is increasing, then f is also increasing.

answer: True

Proof. We want to show that $f(x) < f(y)$ when $x < y$ and we are given that $f_n(x) < f_n(y)$ when $x < y$. I'm going to answer part (f) right now along with this problem. We can see that the convergence only need be pointwise, becuase for any particular choice of x and y, each term $f_n(x)$ and $f_n(y)$ will converge to $f(x)$ and $f(y)$ respesctively. Since each $f_n(x)$ is greater than each $f_n(y)$, the limit of $f_n(x)$ must be less than the limit of $f_n(y)$. In other words, $f(x) < f(y)$. □

(f) Repeat conjecture (e) assuming only pointwise convergence.

answer: See part (e).

6.2.12 Theorem 6.2.6 has a partial converse. Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

answer: Continuity is clear, because g_n is the difference of two continuous functions. Now we show that g_n is decreasing.

$$\begin{aligned} f_n(x) &\leq f_{n+1}(x) \qquad \text{by hypothesis} \\ -f_{n+1}(x) &\leq -f_n(x) \\ f - f_{n+1}(x) &\leq f - f_n(x) \\ g_{n+1} &\leq g_n \end{aligned}$$

Thus g_n is decreasing. Further, since $|f - f_n|$ will become arbitrarily small (because $f_n(x) \to f(x)$), g_n will converge to zero.

(b) Let $\epsilon > 0$ be arbitrary, and define $K_n := \{x \in K : g_n(x) \geq \epsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of compact sets, and use this observation to fiish the argument.

Proof. First I show that the K_n are nested. Let $x \in K_{n+1}$. We want to show that $x \in K_n$. Since $x \in K_{n+1}$, $g_{n+1}(x) \geq \epsilon$. But $g_n \geq g_{n+1} \geq \epsilon$ so $x \in K_n$ as desired. K_n must be closed and bounded in order to be compact. Let x^* be the limit of some sequence $(x_n) \in K$. We want to show that $x^* \in K$, so that K is closed. Since $g_n(x)$ is continuous, $g_n(x_n) \to g(x)$. Each $g(x_n) \geq \epsilon$ however, so it must be true that $g_n(x) \geq 0$. This implies that $x \in K_n$, which implies that K_n is closed. The boundedness arises from the fact that each K_n is a subset of K, which is bounded itself. This means that each K_n is compact.

Recall theorem 3.2.5 which states: If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty. If each K_n was non-empty, then there would be an $\overline{x} \in \bigcap_{n=1}^{\infty} K_n$. But this would mean that g_n does not converge to 0, so there must eventually be an empty K_N. This would imply that $g_n(x) < \epsilon$ for all n greater than N. From this, we can conlude that g_n converges uniformly for all $x \in K$. $\square$

6.2.15 A sequence of functions (f_n) defined on a set $E \subseteq \mathbb{R}$ is called *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$

(a) What is the difference between saying that a sequence of function (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous.

answer: When we say that each f_n is uniformly continuous, we say that we can find a δ, given $\epsilon > 0$, so that when $|x - y| < \delta$, it is true that $|f_n(x) - f_n(y)| < \epsilon$. With equicontinuity, we can find a δ so that $|f_n(x) - f_n(y)| < \epsilon$ ***and*** $|x - y| < \delta$. In other words, with equicontinuity, we can make a choice of δ that does not depend on the particular f_n we are considering.

(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?

answer In order to show that g_n is not equicontinuous, we have to find an ϵ so that we cannot choose an appropriate δ. A convienent choice is $\epsilon = \frac{1}{2}$. It is intuitive that the problem spot is at $x = 1$, which is fixed for all g_n. If we look at a point y, so that $|f_n(1) - f_n(y)| = |1 - y^n| = |1 - y| \cdot |1 + y| < \frac{1}{2}$, then it would have to be true that

$$|1 - y| < \frac{1}{2|y + 1|} \text{ or } 1 < \frac{1}{2|1 + y|} + y$$

This is not possible, so there is no way to choose a δ so that $|1 - y| < \delta$, so g_n is not equicontinuos.

6.2.16 Given $f_n : [0, 1] \to \mathbb{R}$, each f_n is bounded (by some M), and (f_n) is equicontinuous, show that $f_m)$ contains a uniformly convergence subsequence.

(a) Since $\mathbb{Q} = \{r_1, r_2, \dots\}$, and f_n is bounded, we can use problem 6.2.14 to find a subsequence $f_{n_k} =: g_n$ that converges to each rational number in $[0, 1]$. (See part (c) of that problem).

(b) We are given a set $\Lambda = \{r_1, r_2, \dots, r_m\}$ such that $[0, 1] = \bigcup_{r \in \Lambda} V_\delta(r)$, where $|x - y| < \delta$ and $|g_k(x) - g_k(y)| < \frac{\epsilon}{3}$ for some positive ϵ.[4] We want to show that there is an N, so that $|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3}$ for all $s, t \geq N$ and

[4] This is by equicontinuity

$r_i \in \Lambda$. Each g_k is Cauchy since it is uniformly continuous (by Theorem 6.2.5), so we can find an N for any particular r_i in Λ. Because it is given that Λ is finite, we may choose the maximum of these N. If Λ was infinite, we would not be be able to choose such an N.

(c) We can use the conclusion from part (b) with the fact that g_n is equicontinuous to conclude:

$$\begin{aligned} |g_s(x) - g_t(x)| &= |g_s(x) - g_s(r) + g_s(r) - g_t(r) + g_t(r) - g_t(x)| \\ &\le |g_s(x) - g_s(r)| + |g_s(r) - g_t(r)| + |g_t(r) - g_t(x)| \\ &< 3\frac{\epsilon}{3} = \epsilon \end{aligned}$$

Rate of convergence of lefthand Riemann Sums Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous second derivative. Prove that

$$\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=0}^{n-1} f(k/n) = \frac{f(1) - f(0)}{2n} + O\left(\frac{1}{n^2}\right)$$

Proof. We can rewrite $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$ as a sum of integrals,

$$\int_0^{\frac{1}{n}} f(x)\,\mathrm{d}x + \int_{\frac{1}{n}}^{\frac{2}{n}} f(x)\,\mathrm{d}x + \cdots + \int_{\frac{n-1}{n}}^{1} f(x)\,\mathrm{d}x = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,\mathrm{d}x$$

This is useful, becuase it allows to write the difference between $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$ and the lefthand Riemann sum as a sum of integrals. To do this, consider some arbitrary term $\dfrac{1}{n} f\left({}^{k}\!/_{n}\right)$ from the lefthand sums, and note that

$$\int_{\frac{k}{n}}^{\frac{k+1}{k}} f\left(\frac{k}{n}\right)\mathrm{d}x = f\left(\frac{k}{n}\right)\int_{\frac{k}{n}}^{\frac{k+1}{k}} \mathrm{d}x = \frac{1}{n} f\left(\frac{k}{n}\right)$$

Using this fact, we can write $\displaystyle\sum_{k=0}^{n-1} f(k/n)$ as $\displaystyle\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(k/n)\,\mathrm{d}x$. Then, multiplying the difference between the integral and left hand sums by n, we see

$$\begin{aligned} n\left[\int_0^1 f(x)\,\mathrm{d}x - \frac{1}{n}\sum_{k=0}^{n-1} f(k/n)\right] &= n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,\mathrm{d}x - \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(k/n)\,\mathrm{d}x\right] \\ &= n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) - f(k/n)\ \mathrm{d}x\right] \end{aligned}$$

The first order Taylor approximation of some integrand about its lower limit is $f(x) - f\left(\dfrac{k}{n}\right) = \left[f\left(\dfrac{k}{n}\right) - f\left(\dfrac{k}{n}\right)\right] +$

$f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)+R(x)=f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)+\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2$ Using this:

$$
\begin{aligned}
n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)-f(k/n)\ \mathrm{d}x\right] &= n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f'\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)\mathrm{d}x\right]+n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\,\mathrm{d}x\right]\\
&= n\left[\sum_{k=0}^{n-1} f'\left(\frac{k}{n}\right)\int_0^{\frac{1}{n}} u\,\mathrm{d}u\right]+n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\,\mathrm{d}x\right]\\
&= n\left[\sum_{k=0}^{n-1} f'\left(\frac{k}{n}\right)\frac{1}{2n^2}\right]+n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\,\mathrm{d}x\right]\\
&= \frac{1}{2}\sum_{k=0}^{n-1} f'\left(\frac{k}{n}\right)\frac{1}{n}+n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\,\mathrm{d}x\right]\\
&= \frac{1}{2}\int_0^1 f'(x)\,\mathrm{d}x+n\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n}\,\mathrm{d}x\right]\\
\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)-f(k/n)\ \mathrm{d}x &= \frac{f(1)-f(0)}{2n}+\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\,\mathrm{d}x\right]\quad \text{divide by } n
\end{aligned}
$$

All that remains is to show that $\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\,\mathrm{d}x=O\left(\frac{1}{n^2}\right)$ To do this, we note that that we can put a bound on the $f''(c_k)'s$ because $f''(x)$ is continuous over a compact set. We will write $|f''(c_k)|<M$, and therefore we can write

$$
\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left|\frac{1}{2}f''(c_k)(x-\frac{k}{n})^2\right|\mathrm{d}x<\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{2}M(x-\frac{k}{n})^2=M\frac{1}{6n^3}\sum_{k=0}^{n-1}f''(c_k)<\frac{Mn}{6n^3}=\frac{M}{6n^2}\in O\left(\frac{1}{n^2}\right)
$$

□